REPORT ON THE PAPER "THE NUMBER OF INVERSIONS AND THE MAJOR INDEX OF PERMUTATIONS ARE ASYMPTOTICALLY JOINT-INDEPENDENT-NORMAL" BY A. BAXTER AND D. ZEILBERGER

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I have read the whole paper (Third version, Nov. 4, 2010), and found both the result and the method interesting. I believe that the proof is correct, but there are a few points that I would like to see clarified.

(1) Bottom of P2: the description you give of the existence of a (Gaussian) limit law actually resembles the description of a *local* limit law (see for instance Flajolet & Sedgewick, *Analytic combinatorics*, Section IX.9). Even if you carefully explain how da, db and n tend to their respective limits, I am not sure that this can be proved using the method of moments. I suggest you stick to the simplest description, that is, the convergence of the distribution function. You may want to give a reference for the method of moments.

(2) The fact that the moments are polynomials in n is essential in your paper. I would like to see the 'old trick' mentioned on P3 detailed.

(3) P3, below (OO). Could you explain what symmetry makes (OE) and (EO) true before taking the limits? I think I understand why (OE) is true when $s = 0$ (it follows from the fact that the generating function at the top of P2 is self-reciprocal, right?), but we need a symmetry that involves both statistics to prove it for a generic s.

(4) P6, "the leading terms of the FM's and the Mom's are the same". What does it mean? That

$$\mathbb{E}_n(X(X-1)\cdots(X-r+1)Y(Y-1)\cdots(Y-s+1)) \sim \mathbb{E}_n(X^rY^s)$$

for all r and s, where X and Y are respectively the centered inversion number and the centered Major index? Is this really true? and why? For instance

$$\mathbb{E}_n(X(X-1)(X-2)) = -3\mathbb{E}_n(X^2) \quad \text{while} \quad \mathbb{E}_n(X^3) = 0 \quad \text{by symmetry.}$$

Couldn't you say instead that that you apply the method of "factorial" moments, proving the convergence of the factorial moments to those of the normal distribution?

(5) Middle of P7: again, it is not obvious to me that $FM(r,s)(n,i)$, for r and s fixed, is a polynomial in n and i. Neither from the combinatorics, nor from the recurrences. Are you using a complete version of (RecG') and (Gnn')? This would require more explanations.

(6) I was confused by the first sentence of the paragraph "Nice conjectures but what about proofs?". I believe there are two reasons for that:

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- it suggests that the paragraph (only) deals with an alternative way of finding the polynomials $FM(r,s)(n,i)$; as far as I understand, the paragraph is however essential to prove the conjectures.
- since you already used (RecG) and (Gnn) to find many $FM(r,s)(n,i)$, what will change in this alternative approach is not clear.

(7) Finally, now that I have read Dan Romik's report, I agree with him that it would be appropriate to cite this nice formula of Roselle (which I did not know, thank you Dan!). It seems to be one way to answer Point (3) above.

Minor remarks, and some typos

- P2 L1, what does "their" refer to?
- P5, I suggest to give the range of validity of (some of) the identities. For instance, one should certainly not apply (RecF) to $i = n$. At the bottom of this page, it could be worth insisting that $F(n+1, n+1)(p, q)$ is the generating function of permutations of size n (by (Fnn)), which is why you are especially interested in these polynomials.
- P7 L4, "it is still asymptotically normal": explain what "it" refers to. On the next line, can you explain where the value of the average comes from? or give a reference?
- P7, It may be worth defining explicitely the numbers $FM(r,s)(n,i)$ (sorry if I have overlooked their definition!).
- P7, the sentence that contains the expression of $FM(2r, 2s)$ should be preceded by a period, not a comma. At the same place, make clear that the "degree" is the total degree in n and i.
- P8. In (Gnn'), you have kept the term $FM(r-1, s-1)(n-1, i)$ because it may be of the same order as $FM(r,s)(n-1,i)$. But then, shouldn't you have a term $FM(r-1, s-1)(n-1, i)$ in (RecG')? I suggest to replace i by j in (Gnn') since it comes from (Gnn). Also, between (RecG') and (Gnn'), replace Gnn by (Gnn).
- La grande finale (du français !). First line: "the case gives", or "the cases give" (I think...). Recall that you normalize your random variables by σ_n (for a while I had forgotten why you should divide by a power of $FM(0,2)$). Shouldn't there be the parameters (n,i) (or $(n+1, n+1)$) in the last 4 equations? On the next line, "the mixed moments" is repeated. Finally, replace "the normal distribution $e^{-a^2/2-b^2/2}/(2\pi)$" by "the normal distribution of density $e^{-a^2/2-b^2/2}/(2\pi)$".