

Report on Zeilberger's World Cup Problem

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This report considers Zeilberger's World Cup Problem posted on <https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/worldcup.pdf>

1 Problem and result

For an integer $n \geq 3$, let $L_n(t)$ count the nonnegative integer matrices with zero diagonal and with every row sum and every column sum equal to t :

$$L_n(t) = \# \left\{ A = (a_{ij}) \in \mathbb{Z}_{\geq 0}^{n \times n} : a_{ii} = 0, A\mathbf{1} = A^\top \mathbf{1} = t\mathbf{1} \right\}.$$

$$G_n(q) = \sum_{t \geq 0} L_n(t) q^t.$$

Equivalently, this counts loopless directed regular multigraphs in which every labelled vertex has indegree and outdegree t . No symmetry condition is imposed on A .

1.1 Main result

Using the SimpCone algorithm [1] with exact LRQC evaluation and certification as in [2], we computed $G_n(q)$ for every n from 3 through 8. In particular, the computation reaches G_8 , whose complete Ehrhart series is given below. All three independent modular certificates for G_8 pass. At present, this approach appears to have difficulty handling $n \geq 9$.

2 Complete Ehrhart series

We use the following uniform notation:

$$D_n = (n-1)(n-2), \quad s_n = (n-1)(n-3),$$
$$G_n(q) = \frac{H_n(q)}{(1-q)^{D_n}}, \quad H_n(q) = \sum_{j=0}^{s_n} h_{n,j}^* q^j.$$

For each n , the complete coefficient vector of $H_n(q)$ is listed in ascending powers of q .

2.1 G_3

$$G_3(q) = \frac{H_3(q)}{(1-q)^2}.$$

Coefficient vector of $H_3(q)$:

[1]

2.2 G_4

$$G_4(q) = \frac{H_4(q)}{(1-q)^6}.$$

Coefficient vector of $H_4(q)$:

[1, 3, 3, 1]

2.3 G_5

$$G_5(q) = \frac{H_5(q)}{(1-q)^{12}}.$$

Coefficient vector of $H_5(q)$:

[1, 32, 408, 1724, 2765, 1724, 408, 32, 1]

2.4 G_6

$$G_6(q) = \frac{H_6(q)}{(1-q)^{20}}.$$

Coefficient vector of $H_6(q)$:

[1, 245, 22880, 733800, 9847330, 63140236, 209783645, 378349275, 378349275, 209783645, 63140236, 9847330, 733800, 22880, 245, 1]

2.5 G_7

$$G_7(q) = \frac{H_7(q)}{(1-q)^{30}}.$$

Coefficient vector of $H_7(q)$:

[1, 1824, 1199145, 232852850, 17547150075, 620856948600, 11670877850282, 126571440305244, 837957254969529, 3521931467666680, 9653611080929469, 17561784201419010, 21415558464270922, 17561784201419010, 9653611080929469, 3521931467666680, 837957254969529, 126571440305244, 11670877850282, 620856948600, 17547150075, 232852850, 1199145, 1824, 1]

2.6 G_8

$$G_8(q) = \frac{H_8(q)}{(1-q)^{42}}.$$

Coefficient vector of $H_8(q)$:

[1, 14791, 73911915, 86423432893, 33663522704700, 5607920108037548, 470363862820633796, 22178936275290716020, 635311586724884333600, 11696072763886027494936, 144324853681984312695456, 1232343579493807093596080, 7461472646262160852106452, 32641767291337628986522156, 104663610700900943945359492, 248613707850056575120870412, 440814898189449029216178530, 586279441334591901686933654, 586279441334591901686933654, 440814898189449029216178530, 248613707850056575120870412, 104663610700900943945359492, 32641767291337628986522156, 7461472646262160852106452, 1232343579493807093596080, 144324853681984312695456, 11696072763886027494936, 635311586724884333600, 22178936275290716020, 470363862820633796, 5607920108037548, 33663522704700, 86423432893, 73911915, 14791, 1]

3 Computation and certification

3.1 Workflow

$$G_n \xrightarrow{\text{SimpCone}} \{\mathcal{C}_1, \dots, \mathcal{C}_N\} \xrightarrow{\text{LRQC}} G_n(q).$$

SimpCone first decomposes the cone associated with G_n into simplicial cones $\mathcal{C}_i$. LRQC then combines the contributions of these simplicial cones to obtain the exact series $G_n(q)$. Here N is the number of simplicial cones; if N is too large, the computation becomes infeasible.

3.2 SimpCone computation

A final cone is a distinct nonzero simplicial-cone term retained after canonical merging of the SimpCone output.

Table 1. SimpCone decomposition scale and timing

Graph	Merged final output	Time (s)
G_3	1	0.041792
G_4	14	0.046299
G_5	449	0.052214
G_6	25,965	1.111642
G_7	2,331,079	171.586
G_8	325,066,693	10,322.671

In Table 2, an entry $a : c$ means that normalized lattice index a occurs in c final terms. Each row sums to the corresponding final output count.

Table 2. Index distribution of final SimpCone terms

Graph	Index : count
G_3	1 : 1
G_4	1 : 14
G_5	1 : 433, 2 : 16
G_6	1 : 24,612, 2 : 1,321, 3 : 32
G_7	1 : 2,196,661, 2 : 129,009, 3 : 5,333, 4 : 76
G_8	1 : 302,700,582, 2 : 21,229,842, 3 : 1,105,418, 4 : 30,697, 5 : 154

3.3 LRQC computation and CRT reconstruction

The LRQC computation is carried out modulo the following three 61-bit primes:

$$p_A = 2305842973415982001,$$

$$p_B = 2305842962707524241,$$

$$p_C = 2305842957353295361.$$

Table 3. LRQC timings modulo p_A and p_B

Graph	p_A CPU seconds	p_B CPU seconds	$p_A + p_B$	Configuration
G_3	0.000016	0.000053	0.000069	1 thread
G_4	0.000155	0.000157	0.000312	1 thread
G_5	0.013639	0.012982	0.026621	1 thread
G_6	1.793288	1.762547	3.555835	1 thread
G_7	286.935589	291.419190	578.354779	1 thread
G_8	85002.470987	84877.277069	169879.748056	24 + 24 parallel

For G_8 , LRQC is evaluated modulo p_A, p_B, p_C . The three modular results are then combined by the Chinese remainder theorem (CRT) to reconstruct the exact integer coefficients of $G_8(q)$. The p_C computation used 256 slices and 171421.494700 slice CPU seconds; all three modular computations pass.

References

- [1] G. Xin, X. Xu, and Z. Zhang, *A combinatorial simplicial cone decomposition*, arXiv:2501.06691 (2025).
- [2] D. Qiu, G. Xin, and Z. Zhang, *The Ehrhart series of magic squares of orders seven and eight*, arXiv:2607.21338 (2026).