

# REPORT ON ZEILBERGER'S WORLD CUP PROBLEM

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This report considers Zeilberger's World Cup Problem posted on <https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/worldcup.pdf>

## 1. PROBLEM AND RESULT

For an integer  $n \geq 3$ , let  $L_n(t)$  count the nonnegative integer matrices with zero diagonal and with every row sum and every column sum equal to  $t$ :

$$L_n(t) = \# \left\{ A = (a_{ij}) \in \mathbb{Z}_{\geq 0}^{n \times n} : a_{ii} = 0, A\mathbf{1} = A^T\mathbf{1} = t\mathbf{1} \right\}.$$

$$G_n(q) = \sum_{t \geq 0} L_n(t) q^t.$$

Equivalently, this counts loopless directed regular multigraphs in which every labelled vertex has indegree and outdegree  $t$ . No symmetry condition is imposed on  $A$ .

**1.1. Main result.** Using the SimpCone algorithm [1] with exact LRQC evaluation and certification as in [2], we computed  $G_n(q)$  for every  $n$  from 3 through 8. In particular, the computation reaches  $G_8$ , whose complete Ehrhart series is given below. All three independent modular certificates for  $G_8$  pass. At present, this approach appears to have difficulty handling  $n \geq 9$ .

## 2. COMPLETE EHRHART SERIES

We use the following uniform notation:

$$D_n = (n-1)(n-2), \quad s_n = (n-1)(n-3),$$

$$G_n(q) = \frac{H_n(q)}{(1-q)^{D_n}}, \quad H_n(q) = \sum_{j=0}^{s_n} h_{n,j}^* q^j.$$

For each  $n$ , the complete coefficient vector of  $H_n(q)$  is listed in ascending powers of  $q$ .

### 2.1. $G_3$ .

$$G_3(q) = \frac{H_3(q)}{(1-q)^2}.$$

Coefficient vector of  $H_3(q)$ :

[1]

### 2.2. $G_4$ .

$$G_4(q) = \frac{H_4(q)}{(1-q)^6}.$$

Coefficient vector of  $H_4(q)$ :

[1, 3, 3, 1]

2.3.  $G_5$ .

$$G_5(q) = \frac{H_5(q)}{(1-q)^{12}}.$$

Coefficient vector of  $H_5(q)$ :

[1, 32, 408, 1724, 2765, 1724, 408, 32, 1]

2.4.  $G_6$ .

$$G_6(q) = \frac{H_6(q)}{(1-q)^{20}}.$$

Coefficient vector of  $H_6(q)$ :

[1, 245, 22880, 733800, 9847330, 63140236, 209783645, 378349275, 378349275, 209783645, 63140236, 9847330, 733800, 22880, 245, 1]

The above cases agree with Zeilberger's computations.

2.5.  $G_7$ .

$$G_7(q) = \frac{H_7(q)}{(1-q)^{30}}.$$

Coefficient vector of  $H_7(q)$ :

[1, 1824, 1199145, 232852850, 17547150075, 620856948600, 11670877850282, 126571440305244, 837957254969529, 3521931467666680, 9653611080929469, 17561784201419010, 21415558464270922, 17561784201419010, 9653611080929469, 3521931467666680, 837957254969529, 126571440305244, 11670877850282, 620856948600, 17547150075, 232852850, 1199145, 1824, 1]

2.6.  $G_8$ .

$$G_8(q) = \frac{H_8(q)}{(1-q)^{42}}.$$

Coefficient vector of  $H_8(q)$ :

[1, 14791, 73911915, 86423432893, 33663522704700, 5607920108037548, 470363862820633796, 22178936275290716020, 635311586724884333600, 11696072763886027494936, 144324853681984312695456, 1232343579493807093596080, 7461472646262160852106452, 32641767291337628986522156, 104663610700900943945359492, 248613707850056575120870412, 440814898189449029216178530, 586279441334591901686933654, 586279441334591901686933654, 440814898189449029216178530, 248613707850056575120870412, 104663610700900943945359492, 32641767291337628986522156, 7461472646262160852106452, 1232343579493807093596080, 144324853681984312695456, 11696072763886027494936, 635311586724884333600, 22178936275290716020, 470363862820633796, 5607920108037548, 33663522704700, 86423432893, 73911915, 14791, 1]

The formulas for  $G_7$  and  $G_8$  are also obtained by Gabe Vandever using symmetric functions technique. See <https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/DIMACS2026.html>.

### 3. COMPUTATION AND CERTIFICATION

#### 3.1. Workflow.

$$G_n \xrightarrow{\text{SimpCone}} \{\mathcal{C}_1, \dots, \mathcal{C}_N\} \xrightarrow{\text{LRQC}} G_n(q).$$

SimpCone first decomposes the cone associated with  $G_n$  into simplicial cones  $\mathcal{C}_i$ . LRQC then combines the contributions of these simplicial cones to obtain the exact series  $G_n(q)$ . Here  $N$  is the number of simplicial cones; if  $N$  is too large, the computation becomes infeasible.

**3.2. SimpCone computation.** A final cone is a distinct nonzero simplicial-cone term retained after canonical merging of the SimpCone output.

Table 1. SimpCone decomposition scale and timing

| Graph | Merged final output | Time (s)   |
|-------|---------------------|------------|
| $G_3$ | 1                   | 0.041792   |
| $G_4$ | 14                  | 0.046299   |
| $G_5$ | 449                 | 0.052214   |
| $G_6$ | 25,965              | 1.111642   |
| $G_7$ | 2,331,079           | 171.586    |
| $G_8$ | 325,066,693         | 10,322.671 |

In Table 2, an entry  $a : c$  means that normalized lattice index  $a$  occurs in  $c$  final terms. Each row sums to the corresponding final output count.

Table 2. Index distribution of final SimpCone terms

| Graph | Index : count                                                       |
|-------|---------------------------------------------------------------------|
| $G_3$ | 1 : 1                                                               |
| $G_4$ | 1 : 14                                                              |
| $G_5$ | 1 : 433, 2 : 16                                                     |
| $G_6$ | 1 : 24,612, 2 : 1,321, 3 : 32                                       |
| $G_7$ | 1 : 2,196,661, 2 : 129,009, 3 : 5,333, 4 : 76                       |
| $G_8$ | 1 : 302,700,582, 2 : 21,229,842, 3 : 1,105,418, 4 : 30,697, 5 : 154 |

**3.3. LRQC computation and CRT reconstruction.** The LRQC computation is carried out modulo the following three 61-bit primes:

$$\begin{aligned} p_A &= 2305842973415982001, \\ p_B &= 2305842962707524241, \\ p_C &= 2305842957353295361. \end{aligned}$$

Table 3. LRQC timings modulo  $p_A$  and  $p_B$

| Graph | $p_A$ CPU seconds | $p_B$ CPU seconds | $p_A + p_B$   | Configuration    |
|-------|-------------------|-------------------|---------------|------------------|
| $G_3$ | 0.000016          | 0.000053          | 0.000069      | 1 thread         |
| $G_4$ | 0.000155          | 0.000157          | 0.000312      | 1 thread         |
| $G_5$ | 0.013639          | 0.012982          | 0.026621      | 1 thread         |
| $G_6$ | 1.793288          | 1.762547          | 3.555835      | 1 thread         |
| $G_7$ | 286.935589        | 291.419190        | 578.354779    | 1 thread         |
| $G_8$ | 85002.470987      | 84877.277069      | 169879.748056 | 24 + 24 parallel |

For  $G_8$ , LRQC is evaluated modulo  $p_A, p_B, p_C$ . The three modular results are then combined by the Chinese remainder theorem (CRT) to reconstruct the exact integer coefficients of  $G_8(q)$ . The  $p_C$  computation used 256 slices and 171421.494700 slice CPU seconds; all three modular computations pass.

#### REFERENCES

- [1] G. Xin, X. Xu, and Z. Zhang, *A combinatorial simplicial cone decomposition*, arXiv:2501.06691 (2025).
- [2] D. Qiu, G. Xin, and Z. Zhang, *The Ehrhart series of magic squares of orders seven and eight*, arXiv:2607.21338 (2026).

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