Report on the paper "The number of inversions and the major index of permutations are asymptotically joint-independently-normal", by Baxter and Zeilberger

Dan Romik **November 29, 2010**

I was asked to review the probabilistic part of the paper, defined as pages 2–3 (in the version of the paper dated November 5, 2010) and the section titled "A crash course in multivariable enumerative probability". I certify that the material in these sections is correct, except for a couple of minor issues that have no bearing on the overall validity of the main result. The main one is the following: the description of the main result on the bottom half of page 2 is inaccurate (and false if interpreted literally), as it involves the vague differential-like quantities da and db and an unspecified relation between the speeds with which $n \to \infty$ and $da, db \to 0$. The standard way to formalize the statement that a sequence of pairs (X_n, Y_n) of random variables converges to a limiting standard two-dimensional normal distribution is that for all $s, t \in \mathbb{R}$ there is the convergence

$$\Pr(X_n \le s, Y_n \le t) \to \frac{1}{2\pi} \int_{-\infty}^{s} \int_{-\infty}^{t} e^{-(x^2+y^2)/2} \, dy \, dx \quad \text{as } n \to \infty.$$

There are other equivalent ways of saying the same thing, but I don't see why there is a need to confuse readers by using vague and nonstandard language.

A few other minor comments:

1. On the second paragraph of page 4, the authors write that no closed form expression is known for the bivariate generating function of the pair (inv, maj). This is false: such a function is given in D. Knuth's *The Art of Computer Programming, Vol. 3: Sorting and Searching, 2nd Ed.*, solution to Exercise 27 on page 596, where it is attributed to a 1974 paper by D. P. Roselle. In particular, although I haven't tried to do this, asymptotic two-dimensional normality should in principle be provable from such a generating function using standard techniques of asymptotic analysis.

2. On page 3, the fact that the equations (OE) and (EO) are true "before taking the limit" (also mentioned on page 10) is true, but wasn't really that obvious to me and I think deserves a detailed explanation.

3. What the authors call the *correlation* (line 8 of page 3) is usually called the *correlation coefficient*.

4. Footnote number 2 on page 3 refers the reader to section 8.2 of the book *Concrete Mathematics* by Graham, Knuth and Patashnik as an explanation for the fact that the mixed central moment

$$E[(\mathrm{inv}_n - E(\mathrm{inv}_n))(\mathrm{maj}_n - E(\mathrm{maj}_n))]$$

is a polynomial in n of degree ≤ 4. I looked at the reference and didn't find anything there that sheds any light on this matter. Since this polynomial idea is central to the authors' technique, it would be appropriate to add a detailed explanation of why the mixed moments of inv and maj are polynomials and how to bound their degree.