Exploring General Apéry Limits via the Zudilin-Straub t-transform

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Abstract: Inspired by a recent beautiful construction of Armin Straub and Wadim Zudilin, that 'tweaked' the sum of the s^{th} powers of the n-th row of Pascal's triangle, getting instead of sequences of numbers, sequences of rational functions, we do the same for general binomial coefficients sums, getting a practically unlimited supply of Apéry limits. While getting what we call "major Apéry miracles", proving irrationality of the associated constants (i.e. the so-called Apéry limits) is very rare, we do get, every time, at least a "minor Apéry miracle" where an explicit constant, defined as an (extremely slowly-converging) limit of some explicit sequence, is expressed as an Apéry limit of some recurrence, with some initial conditions, thus enabling a very fast computation of that constant, with exponentially decaying error.

Preface: The Major and Minor (but still interesting!) Apéry Miracles

One way that Roger Apéry's [A] [vdP] seminal proof of the irrationality of $\zeta(3)$ could have been discovered, in a *counterfactual world*, was to consider, *out of the blue*, the second-order linear recurrence

$$n^3 u_n \, - \, \left(17n^2+51n+39\right)(2n+3)\; u_{n-1} \, + \, (n-1)^3 u_{n-2} = 0 \quad ,$$

and let a_n and b_n be the solutions of that recurrence with **initial conditions**

$$a_0 = 0, a_1 = 6 \quad ; \quad b_0 = 1, b_1 = 5,$$

then let the computer compute many terms, evaluate $\frac{a_{1000}}{b_{1000}}$ to many decimals, and then use Maple's `identify`, and *lo and behold*, get that it (most probably) equals $\zeta(3)$ (i.e. $\sum_{i=1}^{\infty} \frac{1}{i^3}$). Then, still *empirically* and *numerically*, after rewriting $\frac{a_n}{b_n}$ as $\frac{a'_n}{b'_n}$, where now **both** numerator and denominators are integers (initially b_n were integers, but a_n were not), estimate that there exists a *positive* number δ (about 0.0805) such that

$$|\frac{a'_n}{b'_n} - \zeta(3)| \leq \frac{CONSTANT}{(b'_n)^{1+\delta}} \quad ,$$

that immediately entails (see [vdP]) that $\zeta(3)$ is irrational.

Using the terminology that has now become standard (e.g. the titles of [CS] and [SZ]), we say that $\zeta(3)$ is the *Apéry limit* of the above recurrence, and initial conditions.

The reason that this was a **major** miracle, as explained so eloquently in [vdP], is that while **any** naturally occurring constant that is not **obviously** rational, e.g. the sum of the series $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$, is **definitely** (in the everyday sense of the word) irrational (there are only $\aleph_0$ rational numbers, while there are $2^{\aleph_0}$ real numbers, hence the 'probability' of a real number being rational is a (very small!) 0), it is extremely difficult to (rigorously) **prove** that a *specific* constant is irrational. Witness the fact that, in spite of many attempts, there are still no proofs of the irrationality of the Euler-Mascheroni constant γ (the limit of the partial sums of the harmonic series minus $\log(n)$, or equivalently $-\int_0^{\infty} e^{-x} \log x \, dx$), the Catalan constant $C := \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2}$, $\zeta(5) := \sum_{i=1}^{\infty} \frac{1}{i^5}$ (and more generally $\zeta(2i+1)$, for all $i \geq 2$). The same is also true for $e+\pi$ and $e\,\pi$ (but they can't be **both** rational [proof left to the reader]).

This theme is pursued in [Ze2], [Ze3] and much more recently, in [DKZ], [DZ] and [ZeZu], where the motivation was to discover **irrationality proofs** of other constants. In [DKZ] there were quite a few 'Apéry miracles', alas, most of them were reproved irrationality of constants that were already proved irrational (for example, algebraic numbers or logarithms of them) and the novelty was establishing *explicit irrationality measures*. This is still interesting, but not exciting. We also found few other 'weird' constants given in terms of products of Gamma values at rational numbers, that should yield fully rigorous **first** proofs of **explicit** constants, but since these constants were name-less, it gave us neither fame nor fortune.

But the **minor** Apéry miracle was not **number-theoretical** but rather **numerical-analytical**. Here is an **explicitly** defined constant $\zeta(3) := \sum_{i=1}^{\infty} \frac{1}{i^3}$, in other words the limit of the sequence of rational numbers $\{\sum_{i=1}^{n} \frac{1}{i^3}\}$, that converges **very slowly** to its limit. Realizing it as an *Apéry limit*, i.e. coming up with an explicit **linear recurrence equation with polynomial coefficients**, and two sets of **initial conditions**, for which the limit of the ratios (of the emerging two sequences) converges to that number with an **exponentially decaying error**. This enables one to compute the constant in question to many decimal digits. On the other hand, computing it to that accuracy using the definition would take zillions of years. The purpose of this article is to show how one can produce lots of other 'minor Apéry miracles' where one can express *explicitly* defined constants, whose definition entails very slow convergence, as Apéry limits of recurrences and initial conditions that enables computing these constants with exponentially decaying errors. The key idea is to introduce what we will call the **Zudilin-Straub t-Transform**, that generalizes a construction Armin Straub and Wadim Zudilin discovered [SZ] for the special case of **sums of powers of binomial coefficients**.

But before defining the Zudilin-Straub t-transform, let's define *Apéry limit* more formally, and also introduce the new notion of *Generalized Apéry limit*.

Definition 1: A constant c is an *Apéry limit* if there exists a **homogeneous** linear recurrence with **polynomial coefficients**

$$\sum_{i=0}^{L} p_i(n)\, X(n+i) \;=\; 0 \quad ,$$

and two sets of **initial conditions** $[a_0, a_1, \ldots, a_{L-1}]$ and $[b_0, b_1, \ldots, b_{L-1}]$, such that if $A(n)$ and $B(n)$ are the solutions of that same recurrence with

$$A(0) = a_0 \quad , \ldots , \quad A(L-1) = a_{L-1} \quad ; \quad B(0) = b_0 \quad , \ldots , \quad B(L-1) = b_{L-1} \quad ,$$

then

$$c = \lim_{n\to\infty} \frac{A(n)}{B(n)} \quad .$$

While not part of the definition, it turns out that often, and in all the naturally occurring cases, we also have the following nice feature.

Exponential decay of Error Property

There exist constants $C > 0$ and $\alpha > 1$ such that

$$|\frac{A(n)}{B(n)} - c| \leq \frac{C}{\alpha^n} \quad .$$

Given a recurrence and initial conditions, it is very fast to compute many terms. In fact one only needs **constant memory** (well, linear memory if you go by bit-size) and **linear time** to compute any specific approximation, $\frac{A(n)}{B(n)}$.

Let's introduce a mild extension, that of a *Generalized Apéry limit*.

Definition 2: A constant c is a *Generalized Apéry limit* if there exist two sequences of rational numbers $A(n)$ and $B(n)$ such that

$$c = \lim_{n \to \infty} \frac{A(n)}{B(n)} \quad ,$$

where $A(n)$ and $B(n)$ are solutions of linear recurrences with polynomial coefficients (the first homogeneous, the second inhomogeneous)

$$\sum_{i=0}^{L} p_i(n)\, B(n+i) \, = \, 0 \quad ,$$

subject to initial conditions $B(0) = b_0, \ldots, B(L-1) = b_{L-1}$, and

$$\sum_{i=0}^{L} p_i(n)\, A(n+i) \, = \, C(n) \quad ,$$

subject to initial conditions $A(0) = a_0, \ldots, A(L-1) = a_{L-1}$, where the **right side**, $C(n)$, in turn is a solution of another, (this time homogeneous) linear recurrence equation with polynomial coefficients

$$\sum_{i=0}^{M} q_i(n)\, C(n+i) \, = \, 0 \quad ,$$

subject to some initial conditions $C(0) = c_0, \ldots, C(M-1) = c_{M-1}$.

Note that by using recurrence operators, it is easy to express both $A(n)$ and $B(n)$ as solutions of the **same** homogeneous linear recurrence, so a generalized Apéry limit can always be expressed as an Apéry limit, alas, with the recurrences being of a much higher order, namely $L + M$.

Also note that in order to prove irrationality *à la Apéry*, exponential decay of error does not suffice. After writing the quotients of rational numbers $A(n)/B(n)$ as $\frac{a_n}{b_n}$, where a_n and b_n are **integers**, one needs an inequality of the form,

$$|\frac{a_n}{b_n} - c| \leq \frac{CONSTANT}{b_n^{1+\delta}} \quad ,$$

where $\delta > 0$, yielding an *irrationality measure* $1 + \frac{1}{\delta}$ (see [vdP]). The attempts in [Ze2],[Ze3], [DZ], and [DKZ], to find new (major) Apéry miracles, i.e. irrationality proofs of hopefully new constants, consisted of going *backwards*. Rather than trying to hit the "bull's-eye", one shoots first, and then draws the bull's eye around the bullet hole. Using the Zeilberger, Almkvist-Zeilberger, and multi-Almkvist Zeilberger algorithms (see [DKZ] for references) we generated recurrences for known sequences $B(n)$, then changed the initial conditions, getting a companion sequence $A(n)$, then we computed (very fast, to high accuracy) approximations to the limit of $A(n)/B(n)$. Then we used Maple's `identify` (and our extensions) to **conjecture** an explicit value of the Apéry limit, and hoped to prove it later. But often, neither Maple, nor our extension, was able to *identify* the Apéry limit. In many cases we were able to identify the Apéry limit, but the δ turned out to be negative, so it was useless for proving irrationality. Nevertheless, since we always had $1 + \delta > 0$, we still got **exponentially decaying error**.

In this paper we will forget about our irrationality obsession and only enjoy the exponential decay of error property of Apéry limits (and the generalized version). However here the focus would be to introduce **explicit** constants, defined as a limit of very slowly-converging sequences, and express them as Apéry limits or generalized Apéry limits, enabling computing these constants very fast, to any desired accuracy. In other words, we will do what numerical analysts call *convergence acceleration*, with **very dramatic** acceleration.

The work of Straub and Zudilin that motivated the Zudilin-Straub t-transform

In a recent beautiful article [SZ] (Theorem 1.3 there) (that brilliantly proved some conjectures in the equally beautiful article [CS]) they expressed $\zeta(2j)$ $j = 1, 2, 3\ldots$ (or equivalently π^{2j}) as Apéry limits with **explicit** recurrences and **explicit** initial conditions. The recurrences in question were obtained from the **Zeilberger** [Ze1] (see also [PWZ]) algorithm applied to **sums of binomial powers**, also known as **Franel numbers** (here s is any positive integer):

$$F_n^{(s)} := \sum_{k=0}^{n} \binom{n}{k}^s = \sum_{k=0}^{n} \frac{n!^s}{k!^s (n-k)!^s} \quad ,$$

with $s \geq 2j + 1$. The beauty is that they did it for *infinitely many cases*.

Let's describe what they did. Recall that the **rising factorial**, also called **Pochhammer symbol** (that features in the definition of a **hypergeometric series**), is defined by

$$(x)_n := x\,(x+1) \cdots (x+n-1) \quad .$$

(Note that $(1)_n = n!$.)

The starting point in [SZ] was to consider, instead of the sequence of **integers** $\{F_n^{(s)}\}_{n=1}^{\infty}$, the sequence of **rational functions**, let's call them $\{f_n^{(s)}(t)\}_{n=1}^{\infty}$, defined by

$$f_n^{(s)}(t) := \sum_{k=0}^{n} \frac{n!^s}{(1+t)_k^s (1-t)_{n-k}^s} \quad ,$$

(note that $f_n^{(s)}(0) = F_n^{(s)}$).

Their key idea was to apply the Zeilberger algorithm to the modified sum rather than the original sum, and see what happens. Let's recall some basic definition from **Wilf-Zeilberger algorithmic proof theory** [PWZ] that would also be needed later on, when we will describe our generalization of the Straub-Zudilin [SZ] work.

Definition 3 ([PWZ], p. 64): A discrete function $F(n,k)$ (defined on $\{(n,k)\,|\,0 \leq n,k < \infty\}$) is said to be a *proper hypergeometric term* if it can be written in the form

$$F(n,k) = P(n,k)\,\frac{\prod_{i=1}^{uu}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{vv}(u_i n + v_i k + w_i)!}x^k \quad , \tag{1}$$

in which x is an indeterminate over, say, the complex numbers, and

- P is a polynomial

- the a's, b's, u's, and v's are specific integers, that is to say, they do not contain additional parameters, and

- the quantities uu and vv are finite non-negative, specific integers.

Recall that for any proper hypergeometric term $F(n,k)$ as defined above, the Zeilberger algorithm [Ze1] [PWZ] furnishes, for some non-negative integer L (called the order) , polynomials in n, $p_0(n), \ldots, p_L(n)$, as well as another proper hypergeometric term, $G(n,k)$ called the **certificate** (that furthermore has the property that $G(n,k)/F(n,k)$ is a **rational function** of (n,k)), such that

$$p_0(n)F(n,k) \;+\; p_1(n)F(n+1,k) \;+\; \ldots \;+ p_L(n)F(n+L,k) \;=\; G(n,k+1) - G(n,k) \quad .$$

By summing with respect to k from $k=0$ to $k=n$, we have the immediate corollary that the **hypergeometric sum**

$$f(n) := \sum_{k=0}^{n} F(n,k) \quad ,$$

satisfies the **linear recurrence equation with polynomial coefficients**

$$p_0(n)f(n) \;+\; p_1(n)f(n+1) \;+\; \ldots \;+ p_L(n)f(n+L) \;=\; G(n,n+1) - G(n,0) \quad . \tag{2}$$

Note that in general, the right side is not zero, so one gets an **inhomogeneous** linear recurrence, but whenever the summand is **natural**, for example any **binomial coefficient sum** that contains $\binom{n}{k}$ in it (in particular, of course, for the Franel sequences) the right side vanishes, and one gets that the sum $f(n)$ satisfies a **homogeneous** linear recurrence equation with polynomial coefficients.

This recurrence can be directly extended to the generalized Franel sum as follows. In particular, if

$$p_0(n)F(n,k) + p_1(n)F(n+1,k) + \cdots + p_m(n)F(n+m,k) = G(n,k+1) - G(n,k)$$

holds as an algebraic identity, then replacing k with $t+k$ yields

$$p_0(n)F(n,k+t) + p_1(n)F(n+1,k+t) + \cdots + p_m(n)F(n+m,k+t) = G(n,k+t+1) - G(n,k+t).$$

In other words, $F(n, k+t)$ will "creatively telescope" with the same polynomial recurrence as $F(n,k)$, but the certificate function G will be different.

Now, the generalized Franel sum is

$$f_n^{(s)}(t) := \sum_{k=0}^{n} \frac{n!^s}{(1+t)_k^s (1-t)_{n-k}^s}.$$

Since $(1+t)_k = (t+k)!/t!$ and $(1-t)_{n-k} = (n-(t+k))!/(-t)!$, the summand is precisely the result of replacing k with $k+t$ in the Franel number summand. Therefore the generalized Franel summand will creatively telescope with the same recurrence as the Franel summand itself, but with a different certificate function G that now involves t. Summing over k with the original certificate produces 0, which is where we get the Franel sum recurrences, but summing over k with the new certificate produces $G(n, n+t) - G(n,t)$, which is not 0 but instead a rational function in n and t. This produces the inhomogenuous recurrence

$$p_0^{(s)}(n) f_n^{(s)}(t) + p_1^{(s)}(n) f_{n+1}^{(s)}(t) + \cdots + p_m^{(s)}(n) f_{n+m}^{(s)}(t) = RATIONAL(n,t).$$

Normally we would give up here, but Straub and Zudilin discovered another miracle.

The Straub-Zudilin-Franel miracle

For any $s > 2$, the right side of the inhomogeneous recurrence satisfied by $f_n^{(s)}(t)$ (which is some rational function of n and t) is divisible by t^{s+1} if s is odd and by t^s if s is even. In particular, the coefficient on t^r in the right-hand side is zero for $r < s$. It follows that taking coefficients on t^r in the equation

$$p_0^{(s)}(n) f_n^{(s)}(t) + p_1^{(s)}(n) f_{n+1}^{(s)}(t) + \cdots + p_m^{(s)}(n) f_{n+m}^{(s)}(t) = RATIONAL(n,t)$$

produces a sequence in n which satisfies the same equation, but with a 0 on the right-hand side. That is, the sequence satisfies the **very same** *homogeneous* linear recurrence satisfied by the Franel sequence $F_n^{(s)}$, only with different initial conditions. Hence the limit of the ratios of these new sequences with the Franel numbers $F_n^{(s)}$ is an Apéry limit of *some constant*.

Can we describe this constant directly? Yes we can!

Let's rewrite the sum $f_n^{(s)}(t)$ as follows:

$$f_n^{(s)}(t) := \sum_{k=0}^{n} \frac{n!^s}{(1+t)_k^s (1-t)_{n-k}^s} = \sum_{k=0}^{n} \frac{n!^s}{k!^s (n-k)!^s} \cdot \left(\frac{k!^s (n-k)!^s}{(1+t)_k^s (1-t)_{n-k}^s} \right) \quad .$$

Let's give the coefficient of t^r in the Taylor expansion (about $t = 0$) of $\frac{k!^s (n-k)!^s}{(1+t)_k^s (1-t)_{n-k}^s}$ a name.

Definition 4: $c_r^{(s)}(n,k)$ is the coefficient of t^r in the Taylor expansion of the rational function of t $\frac{k!^s (n-k)!^s}{(1+t)_k^s (1-t)_{n-k}^s}$.

Interesting consequences of the Straub-Zudilin-Franel miracle

Now fix a power $s > 2$ and $r < s+1$, and let $B(n)$ be the Franel sequence and $A(n)$ the sequence of the coefficients of t^r in the Taylor expansion of $f_n^{(s)}(t)$. Then

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^s \quad ,$$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^s c_r^{(s)}(n,k) \quad .$$

Now consider the Apéry limit

$$\lim_{n\to\infty} \frac{A(n)}{B(n)} = \frac{\sum_{k=0}^{n} \binom{n}{k}^s c_r^{(s)}(n,k)}{\sum_{k=0}^{n} \binom{n}{k}^s} \quad .$$

Note that the left side is a certain *Apéry limit* with the Franel recurrence and appropriate initial conditions (hence the limit can be computed very fast, with exponentially-decaying error). On the other hand the right side is a *weighted-average* of the $n+1$ numbers

$$\{ c_r^{(s)}(n,k) \,|\, 0 \le k \le n \} \quad ,$$

with *weights* $\{ \binom{n}{k}^s \,|\, k = 0, \dots, n \}$. Since (recall the *Central Limit Theorem*, see De Bruijn's classic, section 3.4, for details) most of the weight is in the middle, we have that the above Apéry limit has an explicit description, namely

$$\lim_{n\to\infty} c_r^{(s)}(n, \lfloor n/2 \rfloor) \quad .$$

For any *specific* r, one can express $c_r^{(s)}(n,k)$ in terms of partial sums of the harmonic series or partial sums of $\zeta(i)$ (i.e. $\sum_{k=1}^{n} \frac{1}{k^i}$). As r gets larger, things get complicated but our Maple package `Zudilin.txt` can handle it easily (it is implemented by procedure `CrL(A,r,L)`). In particular, for $r = 2$, things are still fairly easy as the following lemma explains. The proof involves elementary manipulations with power series, fully automated in our Maple package, but would be cumbersome to reproduce here.

Lemma 1:

$$c_2^{(s)}(n,k) = \frac{s}{2} \left(\sum_{i=1}^{k} \frac{1}{i^2} + \sum_{i=1}^{n-k} \frac{1}{i^2} \right) + \frac{s^2}{2} \left(-\sum_{i=1}^{k} \frac{1}{i} + \sum_{i=1}^{n-k} \frac{1}{i} \right)^2 \quad .$$

Note that in particular, the Apéry limit for the case $r=2$ in the Straub-Zudilin t-version of Franel equals $s\,\zeta(2)$.

For each specific integer s, (and also for each specific even integer r) we can prove that (for sufficiently large s) you get $\zeta(r)$, as is proved, in general, using human ingenuity, in [SZ].

Generalized Franel Sums

The Zeilberger algorithm, just as easily, can find a homogeneous linear recurrence equation with polynomial coefficients for the generalized Franel sum

$$\sum_{k=0}^{n} \binom{n}{k}^{s} a^k \quad ,$$

for *any* positive rational number a (or for that matter even for symbolic a). It turns out that for all the s that we tried, the same miracle, i.e. that the right side of the linear **inhomogeneous** recurrence satisfied by

$$\sum_{k=0}^{n} \frac{n!^s}{(1+t)_k^s (1-t)_{n-k}^s} \cdot a^k \quad ,$$

is divisible by t^r for $r \leq s$. It follows that the first s Taylor coefficients vanish. We are sure that this is provable in general using the method of [SZ], but we leave it to the interested reader.

Let us now state the following result.

Theorem 1: For any positive rational number a, and for $s \geq 3$, and for $1 \leq r < s$ if s is odd, and for $1 \leq r \leq s-1$ if s is even. Let α be the such that, as $n \to \infty$, $k = \alpha n$ maximizes the summand

$$\binom{n}{k}^{s} a^k$$

(this can be easily found by computing the ratio of consecutive terms, setting it equal to 1 and solving for α as $n \to \infty$, it is implemented in our Maple package by procedure `FindMaxk(F,n,k)`). (Note that α is always an algebraic number, that our Maple package can always find in each case.) Let $B(n)$ be $\sum_{k=0}^{n} \binom{n}{k}^{s} a^k$ and let $A(n)$ be solution of the same recurrence, of order L, say, that is satisfied by $B(n)$ by the Zeilberger algorithm, but with the initial conditions extracted from the coefficients of t^r in $\sum_{k=0}^{n} \left(\frac{n!}{(1+t)_k (1-t)_{n-k}}\right)^{s} a^k$, for $n = 0, \ldots, n = L-1$, then the sequence $\frac{A(n)}{B(n)}$ converges **with exponentially decaying error** to the constant

$$\lim_{n\to\infty} c_r^{(s)}(n, \lfloor \alpha n \rfloor) \quad .$$

Disclaimer: We took the liberty of calling the above statement "Theorem", and we are sure that it is correct, but we did not work out the full details of the proof. Wadim Zudilin informed us that indeed he strongly believes that the full proof should follow along the lines described in [deB], section 3.4. and [SZ].

The proof of the *exponential decay of error* property, *in general* for *all* such binomial coefficients sums, and more generally, for any binomial coefficient sum with **positive** summand, follows the same lines as the special case of Apéry's original proof in [vdP] (p. 196), and is omitted here. Note that for any **specific** binomial coefficient sum, it is proved **fully rigorously**, *ab initio*, by the computer.

Note that trying to evaluate this limit from the definition would take for ever, since the convergence is so slow. So we can get lots of *minor Apéry miracles.*

The output files `https://sites.math.rutgers.edu/~zeilberg/tokhniot/oZudilin1.txt` and `https://sites.math.rutgers.edu/~zeilberg/tokhniot/oZudilin2.txt` contain many examples.

Let's explain why α is algebraic. The summand ${n \choose k}^s a^k$ is clearly **unimodal**, hence it is maximal for that value of k when the ratio of consecutive terms satisfy

$$\left(\frac{n-k+1}{k}\right)^s a \leq 1 < \left(\frac{n-k}{k+1}\right)^s a$$

Writing $k = \alpha n$, and remembering that $n \to \infty$, we have

$$(1-\alpha)^s a \,=\, 1 \quad ,$$

and this an algebraic equation for α for any specific positive integer s and any specific positive rational number $a \geq 1$.

The Zudilin-Straub t-Transform

The beautiful construction in [SZ] was obtained by replacing $k!$ by $(1+t)_k$ and $(n-k)!$ by $(1-t)_{n-k}$. This naturally leads to

Definition 5: The Zudilin-Straub t-transform of the proper hypergeometric term given in (1) is

$$\hat{F}(n,k;t) = P(n,k+t)\,\frac{\prod_{i=1}^{uu}(b_i t+1)_{a_i n+b_i k+c}}{\prod_{i=1}^{vv}(v_i t+1)_{u_i n+v_i k+w_i}} x^k \quad .$$

It is immediate to see that the recurrence obtained via the Zeilberger algorithm applied to the Zudilin-Straub t-transform of any proper hypergeometric term is the same as the original, **except** that the right side is not *zero*, i.e., since the right side of Equation (2) is no longer 0, the linear recurrence is *inhomogeneous* Unfortunately, in general, the Franel-Straub-Zudilin miracle does not occur, but it is easy to see that for any **specific** positive integer r the coefficient of t^r in the Taylor expansion of the right side is still P-recursive in n (i.e. satisfies a linear recurrence equation with polynomial coefficients), and a recurrence for it can be algorithmically obtained, either by the 'holonomic machine' ([ApaZ] [K]), or by 'guessing' that can be made fully rigorous using the general theorems of [ApaZ]. In this way we can get lots of *generalized Apéry limits* describing constants defined as limits of explicit sequences that converge very slowly. Rather than stating the formal theorem we refer the reader to the output files given in the front of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/zudilin.html` .

In particular, we recommend the following output files

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oZudilin1.txt`

for many examples of Theorem 1, and the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oZudilin3a.txt` , and

`https://sites.math.rutgers.edu/žeilberg/tokhniot/oZudilin5.txt` ,

for examples where the recurrence equation for $B(n)$ is inhomogeneous

The Maple package `Zudilin.txt` implements everything. Once you download it to your computer (that has Maple), load it to a Maple session with `read 'Zudilin.txt'`. To get a list of the main functions, type `ezra();`, and to get help with a specific function, type `ezra(FunctionName)`. For example, to get help with procedure `ZT` (that implements the Zudilin-Straub t-transform) type `ezra(ZT);`. Enjoy!

Conclusion

While major Apéry miracles are few and far between, thanks to the **Zudilin-Straub** t-transform, we can obtain many minor Apéry miracles. The front of this article contains numerous examples, but using our Maple package, readers can generate many new ones.

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