YAY FOR DETERMINANTS!

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ABSTRACT. In this *case study*, we hope to show why Sheldon Axler was not just wrong, but *wrong*, when he urged, in 1995: "Down with Determinants!". We first recall how determinants are useful in enumerative combinatorics, and then illustrate three versatile tools (Dodgson's condensation, the holonomic ansatz and constant term evaluations) to operate in tandem to prove a certain intriguing determinantal formula conjectured by the first author.

1. The Joy of Determinants

In 1995, Sheldon Axler published an article [2] with a very catchy title (that being determinantal lovers, we *love to hate*): 'Down with Determinants!' This admittedly well-written paper was well received and even won the prestigious Lester Ford award for that year. Let's quote the first paragraph.

"Ask anyone why a square matrix of complex numbers has an eigenvalue, and you'll probably get the wrong answer, which goes something like this: The characteristic polynomial of the matrix which is defined via determinants has a root (by the fundamental theorem of algebra); this root is an eigenvalue of the matrix. What's wrong with that answer? It depends upon determinants, that's what. Determinants are difficult, non-intuitive, and often defined without motivation. As we'll see, there is a better proof one that is simpler, clearer, provides more insight, and avoids determinants."

Axler then goes on to describe a *determinant-less* approach to linear algebra, that while very elegant, is **not** to our liking. It is way too *abstract* for our taste. We believe that one of the ills of undergraduate mathematics instruction is excess abstraction. We love determinants because they are so **concrete**.

In our humble opinion, determinants are **easy**, **intuitive**, and can be easily defined with great motivation. In fact, it is a straightforward extension of the good-old factorial function $n!$. Recall that $n!$ **counts** the number of permutations of the set $\{1, \ldots, n\}$. In other words

$$n! = \sum_{\pi \in S_n} 1 \, .$$

Now for any permutation $\pi = (\pi_1, \ldots, \pi_n)$ define a *weight*

$$\mathrm{Weight}(\pi) := \mathrm{sign}(\pi) \, a_{1,\pi_1} \cdots a_{n,\pi_n} \, ,$$

and then the determinant is *simply* the weight-enumerator of the symmetric group:

$$\det(a_{ij}) = \sum_{\pi \in S_n} \mathrm{Weight}(\pi) \, .$$

Note that our definition is **combinatorial**, and indeed they come up so often in tough enumeration problems, where it is already a challenge to express the quantity of interest as an **explicit** (symbolic) determinant, and then an often bigger challenge is to **evaluate** it in **closed form**. In this *case study* we will discuss various ways to tackle such determinants, by focusing on one 'hard nut' that came up in MathOverflow.

2. An Intriguing Determinant

The discussion in this article is motivated by a question on MathOverflow [4] posed by Johann Cigler wherein he asked for a simple direct proof of

$$\det A_{n,m} := \det\left[\binom{2m}{j-i+m} - \binom{2m}{m-i-j-1}\right]_{i,j=0}^{n-1} = \prod_{i\le j}^{1,m-1} \frac{2n+i+j}{i+j}.$$

The expression on the right-hand side of the identity is connected to a host of interesting combinatorial interpretations (see references [9], [10], [8], [19] and [20]).

Our journey began with a generalization (true to Pólya's dictum: *generalize to trivialize*). Introduce a new parameter x and a matrix given by

$$T_{n,m}(x) := \left[\binom{x+m}{j-i+m} - \binom{x+m}{m-i-j-1}\right]_{i,j=0}^{n-1}.$$

After enough experimentation, we managed to guess a closed-form evaluation for its determinant.

Conjecture 1. *We have*

$$\det T_{n,m}(x) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{(x+i-j)(x+2i+j-2)}{(x+2i-j)(i+j-1)}.$$

None of our initial attempts at proving this claim worked.

Remark 1. In the meantime, we made the following curious observation. If

$$\varphi_n(y) := \frac{(1+y^n-y^{n+1}+y^{2n})(1-y^n)^2}{y^{n-1}(1+y)(1-y)^2} \in \mathbb{Z}[y,y^{-1}]$$

then

$$\det T_{n,n}(x) = n!^{-n} \prod_{i=0}^{n-1} \binom{n+i}{i}^{-1} \cdot \prod_{s=1-n}^{3n-2} (x+s)^{CT_y(y^{-s}\varphi_n(y))};$$

where $CT_y\, F(y)$ denotes the *constant term* of a Laurent polynomial $F(y)$.

3. Enter Dodgson

A relatively simple choice for conjecturing and proving explicit determinant evaluations is a method inspired by *Dodgson's condensation* [22], that goes back to Jacobi and Desanot, and popularized by the Reverend Charles L. Dodgson (aka Lewis Caroll). However, $\det T_{n,m}(x)$ did not yield to this at all. It was necessary to change the approach. Our first breakthrough was realized in the form of discovering a new matrix

$$D_n(a,b) := \left[\binom{2i+2a}{i-j+a-b} - \binom{2i+2a}{i-j+a-b-1}\right]_{i,j=0}^{n-1}$$

which, happily, is amenable to the present technique.

Theorem 3.1. *We have the determinantal evaluation*

$$\det D_n(a,b) = \prod_{i=1}^{n}\prod_{j=1}^{a-b} \frac{(a+b+i-j)(a+b+2i+j-2)}{(a+b+2i-j)(i+j-1)}.$$

Proof. Denoting the determinant by $M_n(a,b)$, Dodgson's condensation states that

$$M_n(a,b) = \frac{M_{n-1}(a,b)M_{n-1}(a+1,b+1) - M_{n-1}(a+1,b)M_{n-1}(a,b+1)}{M_{n-2}(a+1,b+1)}.$$

Therefore, we only need to verify (routinely) that the right-hand side does the same. To conclude, simply check both sides agree, say, when $n = 1$ and $n = 2$. □

For reasons that will become clearer soon, let's reformulate the above result.

Corollary 3.2. *We have*

$$\begin{aligned}\det B_{n,m}(x) :&= \det\left[\binom{x+m+2i}{i-j+m} - \binom{x+m+2i}{i-j+m-1}\right]_{i,j=0}^{n-1}\\ &= \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{(x+i-j)(x+2i+j-2)}{(x+2i-j)(i+j-1)}.\end{aligned}$$

Proof. This follows from Theorem 2.1 with $a = \frac{x+m}{2}$ and $b = \frac{x-m}{2}$ in $D_n(a,b)$. □

Note that the *right hand sides* of the (already proved!) Corollary 3.2 and the (still unproved) Conjecture 1 are the same, but why is it useful? We will see in the next section that the **left hand sides** are the same, and, hence Conjecture 1 would follow from the *transitivity* of the = relation: $A = B$ and $B = C \Rightarrow A = C$.

This method was famously used by one of us (DZ) to prove the Mills-Robbins-Rumsey *alternating sign matrix conjecture* [26] (see also [3]). In that article the author first found a (complicated) constant term expression for the desired quantity (the number of alternating sign matrices) and *another* (almost as complicated) constant term expression for another quantity (*totally symmetric self-complementary plane partitions*), already proved by guru George Andrews to be equal to the desired expression. He then worked very hard to prove that these two constant term expressions are the same, and finally, by taking a*free ride* from Andrews' previously proved enumeration, and using the above '*transitivity* of =', gave the **first** proof of this notorious conjecture.

In the next section we will use this methodology, but things will be much simpler, since we will prove that the left sides of Conjecture 1 and Corollary 3.2 are given by the *same* constant-term expression, so unlike [26], we will not have to work hard to prove that $A = B$, since A is **exactly** B.

4. Enter Constant Term Evaluations

As promised at the end of the last section, in this section, we describe the connection between the two matrices $B_{n,m}(x)$ and $T_{n,m}(x)$ from the preceding sections.

Theorem 4.1. *It holds true that* $\det T_{n,m}(x) = \det B_{n,m}(x)$*, and hence so does Conjecture 1.*

Proof. Denote the *constant term* of the Laurent series expansion of $F(t)$ by $CT_t F$. In view of this, the entries of $T_{n,m}(x)$ can be expressed as a constant term:

$$\binom{x+m}{j-i+m} - \binom{x+m}{m-i-j-1} = CT_t \left(1+\frac{1}{t}\right)^{x+m} t^m (t^{j-i} - t^{-i-j-1}).$$

The entries in the i^{th} row of the matrix $T_{n,m}(x)$ are (we need a different variable for each row, let's call it t_i - this does not change the values)

$$CT_{t_i} \left(1+\frac{1}{t_i}\right)^{x+m} t_i^m (t_i^{j-i} - t_i^{-i-j-1}).$$

Let $\vec{t} = (t_0, t_1, \ldots, t_{n-1})$. We apply multi-linearity of determinants to obtain

$$\begin{aligned}
\det T_{n,m}(x) &= \det \left[CT_{t_i} \left(1+\frac{1}{t_i}\right)^{x+m} t_i^{m-i} \left(t_i^j - t_i^{-j-1}\right) \right] \\
&= CT_{\vec{t}} \left[\prod_{i=0}^{n-1} (1+t_i)^{x+m} \cdot t_i^{-x-i-\frac{1}{2}} \cdot \det \left(t_i^{j+\frac{1}{2}} - t_i^{-j-\frac{1}{2}}\right) \right] \\
&= CT_{\vec{t}} \left[\prod_{i=0}^{n-1} \frac{(1+t_i)^{x+m}(t_i-1)}{t_i^{x+n+i}} \cdot \prod_{i<j}^{0,n-1} (t_i - t_j)(1 - t_i t_j) \right];
\end{aligned}$$

where we have utilized [14, Lemma 2]. Let us *symmetrize* the right-hand side (by averaging over the images of the symmetric group $\mathfrak{S}_n$, using the fact that the functional CT is unaffected by permuting the variables, since **a constant is always a constant**, this was called the *Stanton-Stembridge trick* in [26]). Note that only the product of two factors $\prod_{i=0}^{n-1} t_i^{-i} \cdot \prod_{i<j} (t_i - t_j)$ will be affected to introduce a new Vandermonde:

$$\begin{aligned}
&\det T_{n,m}(x) \\
&= \frac{1}{n!} CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[\frac{(1+t_i)^{x+m}(t_i-1)}{t_i^{x+n}} \right] \cdot \prod_{i<j}^{0,n-1} (t_i - t_j)(t_j^{-1} - t_i^{-1})(1 - t_i t_j) \right\} \\
&= \frac{1}{n!} CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[\frac{(1+t_i)^{x+m}(t_i-1)}{t_i^{x+2n-1}} \right] \cdot \prod_{i<j}^{0,n-1} (t_i - t_j)^2 (1 - t_i t_j) \right\}.
\end{aligned}$$

Now, repeat the constant term extraction on the determinant from the previous section. The outcome of this procedure is:

$$\begin{aligned}
&\det B_{n,m}(x) \\
&= \det \left[CT_{t_i} \left(1+\frac{1}{t_i}\right)^{x+m+2i} t_i^m (t_i^{i-j} - t_i^{i-j-1}) \right] \\
&= CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[\left(1+\frac{1}{t_i}\right)^{x+m+2i} t_i^{m+i-1}(t_i - 1) \right] \prod_{i<j}^{0,n-1} (t_j^{-1} - t_i^{-1}) \right\} \\
&= CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[\frac{(1+t_i)^{x+m}(t_i-1)}{t_i^{x+n}} \right] (2 + t_i + t_i^{-1})^i \prod_{i<j}^{0,n-1} (t_i - t_j) \right\}
\end{aligned}$$

$$= \frac{1}{n!}\, CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[\frac{(1+t_i)^{x+m}(t_i-1)}{t_i^{x+n}} \right] \prod_{i<j}^{0,n-1} (t_i - t_j)\left(t_j + t_j^{-1} - t_i - t_i^{-1}\right) \right\}$$
$$= \frac{1}{n!}\, CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[\frac{(1+t_i)^{x+m}(t_i-1)}{t_i^{x+2n-1}} \right] \prod_{i<j}^{0,n-1} (t_i - t_j)^2 (1 - t_i t_j) \right\}.$$

Based on our analysis shown above, both determinants $\det T_{n,m}(x)$ and $\det B_{n,m}(x)$ are equal to the same constant term

$$\frac{1}{n!}\, CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[(1+t_i)^{x+m}(t_i-1)t_i^{-x-2n+1} \right] \prod_{i<j}^{0,n-1} (t_i - t_j)^2 (1 - t_i t_j) \right\}.$$

Hence $\det T_{n,m}(x)$ and $\det B_{n,m}(x)$ must be equal to each other, as claimed. $\square$

In particular, we thus recover the determinant of Cigler as stated in Section 1:

Corollary 4.2. (*The case $x = m$*) *We have*

$$\det \left[\binom{2m}{j-i+m} - \binom{2m}{m-i-j-1} \right]_{i,j=0}^{n-1} = \prod_{1 \le i \le j \le m-1} \frac{2n+i+j}{i+j}.$$

Proof. The specialization $x = m$ in the formula for $\det T_{n,m}(x)$ results in

$$\det T_{n,m}(m) = \prod_{1 \le i \le j \le m-1} \frac{2n+i+j}{i+j}$$

after some simplification, which is $\det A_{n,m}$. $\square$

In addition, we believe to have uncovered a *new* constant term evaluation which appears not to fit any of the existing formulas in the literature.

Corollary 4.3. *We have*

$$CT_{\vec{t}} \left\{ \prod_{i=0}^{n-1} \left[\frac{(1+t_i)^{x+m}(t_i-1)}{t_i^{x+2n-1}} \right] \prod_{i<j}^{0,n-1} (t_i - t_j)^2 (1 - t_i t_j) \right\}$$
$$= n! \cdot \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{(x+i-j)(x+2i+j-2)}{(x+2i-j)(i+j-1)}.$$

Proof. This is a consequence of Corollary 2.2 and the proof of Theorem 3.1. $\square$

5. Enter the Holonomic Ansatz

In this section, we use an automated method called the *holonomic ansatz* due to the third author [24], and extended by the second author and collaborators [5, 11, 13] to discover and prove many new deep determinant identities, which will provide yet another proof of Conjecture 1. Note: we also reindex $1 \le i, j \le n$ instead of $0 \le i, j \le n-1$. Let us recall the matrix $A_n := A_n(m, x) := (a_{i,j})_{1 \le i,j \le n}$ whose entries $a_{i,j}$ are given as follows:

$$a_{i,j} := a_{i,j}(m, x) := \binom{m+x}{m-i+j} - \binom{m+x}{m-i-j+1}.$$

Denote by $A_n^{(i,j)}$ the matrix A_n with the i^{th}-row and the j^{th}-column being removed. Then we define

$$c_{n,j} := c_{n,j}(m,x) := (-1)^{n+j}\,\frac{\det A_n^{(n,j)}}{\det A_{n-1}}. \tag{1}$$

Employing an ansatz with undetermined coefficients ("guessing"), we find plausible recurrence relations that are conjecturally satisfied by the sequence $c_{n,j}$, which suggest that it is holonomic with rank 2:

$$\begin{aligned}
&j(j-n-1)(m+n)(2n+x-1)(2n+x)c_{n+1,j}+\\
&(2j-1)n(j-m)(2n+x)(j-n-x+1)c_{n,j+1}+\\
&n(j+n+x-1)(4j^2n-(m^2+4n^2-m)j\\
&\qquad -2mn+(2j^2-4jn-m)x-jx^2)c_{n,j}=0, \qquad \text{and}\\
&j(j-m+1)(j+n+1)(j-n-x+2)c_{n,j+2}\\
&\ +(2j^4+4j^3-j^2m^2-2j^2n^2-2j^2nx+2j^2n-j^2x^2+2j^2x+2j^2-jm^2\\
&\quad -2jn^2-2jnx+2jn-jx^2+2jx-mn^2-mnx+mn)c_{n,j+1}\\
&\ +(j+1)(j+m)(j-n)(j+n+x-1)c_{n,j}=0.
\end{aligned}$$

For the purpose of guessing, one has to evaluate $c_{n,j}$ for concrete values of n and j; in this particular example it suffices to consider $1 \leq j \leq n \leq 15$. For a concrete integer n, the vector $(c_{n,j})_{1\leq j\leq n}$ can be determined by computing the kernel of the matrix $(a_{i,j})_{1\leq i<n, 1\leq j\leq n}$. Note that this kernel computation appears to be challenging, because the matrix entries are not rational functions in the parameters m and x. To overcome this problem, and also for efficiency reasons, we employ the evaluation-interpolation technique: when m and x are substituted by concrete integers, then the matrix entries turn into integers, and the recurrences for $c_{n,j}$ can be easily guessed. Performing the same computation for different choices of m and x allows one to reconstruct recurrences with symbolic m and x, as given above.

The shape of the above recurrences, i.e., their support and their leading coefficients, implies that the following three initial conditions are sufficient to define a unique bivariate sequence: $c_{1,1} = 1$, $c_{1,2} = 0$, $c_{2,2} = 1$. From now on, let $c_{n,j}$ denote this unique bivariate sequence that is defined by unrolling the recurrences, starting from the given initial conditions. We want to prove that this conjectured definition of $c_{n,j}$ agrees with its original definition (1). For this purpose it suffices to establish the two identities:

$$c_{n,n} = 1 \qquad (n \geq 1), \tag{2}$$

$$\sum_{j=1}^{n} a_{i,j}c_{n,j} = 0 \qquad (1 \leq i < n), \tag{3}$$

which can be achieved by holonomic closure properties and creative telescoping, respectively. We use the Mathematica package HolonomicFunctions [12]. Observe that the closure property "integer-linear substitution" implies that $c_{n,n}$ satisfies a univariate recurrence of order at most 2. We compute this recurrence and show that it admits a constant solution. Then the two initial values $c_{1,1} = c_{2,2} = 1$ imply that $c_{n,n} = 1$ for all $n \geq 1$.

In order to prove (3), we split the sum into two parts:

$$s_{i,n} := \sum_{j=1}^{n} a_{i,j} c_{n,j} = \sum_{j=1}^{n} \binom{m+x}{m-i+j} c_{n,j} - \sum_{j=1}^{n} \binom{m+x}{m-i-j+1} c_{n,j}.$$

Now creative telescoping can be employed to derive a set of recurrences for each of the sums on the right-hand side, which can be combined to a set of recurrences that is satisfied by the whole expression $s_{i,n}$. These computations are far from trivial (it took about two hours of CPU time) and the recurrences are too big to be displayed here. It can easily be checked that $s_{i,n} = 0$ for a few concrete small integers i and n, and the computed recurrences then imply that $s_{i,n} = 0$ for all $1 \le i < n$.

Finally, we investigate the sum

$$t_n := t_n(m,x) := \sum_{j=1}^{n} a_{n,j} c_{n,j} = \sum_{j=1}^{n} \binom{m+x}{m-n+j} c_{n,j} - \sum_{j=1}^{n} \binom{m+x}{m-n-j+1} c_{n,j}.$$

Another two applications of creative telescoping exhibit that both sums on the right-hand side satisfy the same recurrence, which therefore is also valid for their difference t_n:

$$\begin{aligned}&(m+n)(2n+x-1)(2n+x+1)(2n+x)^2(m-n-x)t_{n+1}\\ &\quad + n(n+x)(m-2n-x-1)(m-2n-x)(m+2n+x-1)(m+2n+x)t_n = 0.\end{aligned}$$

Since this is a first-order recurrence, a closed form for t_n is immediately obtained in terms of *Euler's gamma function*:

$$t_n = \frac{\Gamma(n)\,\Gamma(n+x)\,\Gamma(2n+x-m)\,\Gamma(2n+x+m-1)}{\Gamma(n+m)\,\Gamma(n+x-m)\,\Gamma(2n+x)\,\Gamma(2n+x-1)}.$$

From the definition (1) of $c_{n,j}$ it follows that

$$t_n = \frac{\det A_n}{\det A_{n-1}},$$

which yields the desired closed form of our determinant:

$$\det A_n = \prod_{i=1}^{n} t_i = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{t_i(j,x)}{t_i(j-1,x)} = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{(x+i-j)(x+2i+j-2)}{(x+2i-j)(i+j-1)}. \qquad \square$$

The special case $x = m$: we point out something very pleasant happening here. The entries $a_{i,j}$ of the matrix $T_{n,m}(m)$ are given by

$$a_{i,j} = \binom{2m}{m-i+j} - \binom{2m}{m-i-j+1},$$

and the corresponding determinant by

$$b_n := \det T_{n,m}(m) = \prod_{1 \le i \le j \le m-1} \frac{2n+i+j}{i+j}.$$

The key difference: this time, we are able to construct an *explicit* double-indexed sequence $\mathbf{c}$ according to $c_{n,n} = 1$ and for $1 \le j < n$,

$$c_{n,j} = \frac{(-1)^{n-j}\binom{2n-1}{n-j}\binom{n+m-j-1}{n-j}}{\binom{2n+m-2}{n-j}}.$$

Then, Zeilberger's holonomic ansatz rules supreme once the following system of equations are proven to hold true:

$$\begin{cases} c_{n,n} = 1 & n \geq 1 \\ \sum_{j=1}^{n} a_{i,j}\, c_{n,j} = 0 & 1 \leq i \leq n-1 \\ \sum_{j=1}^{n} a_{n,j}\, c_{n,j} = \frac{b_n}{b_{n-1}} & n \geq 1. \end{cases}$$

But, these are directly justified with the help of *Zeilberger's algorithm* [25]. We will not pursue this matter because we have already proven Conjecture 1 (see above), in its full generality.

6. $T_{n,m}(x)$ Finds its Twin

Suppressing the parameters a and b, and defining $u_k := \binom{a+b}{a-k}$, the matrix $T_{c,a}(b)$ is nothing but the difference of a Toeplitz and a Hankel matrix:

$$T_{c,a}(b) = [u_{i-j} - u_{i+j+1}].$$

By analogy, we may introduce its natural companion as a sum of a Toeplitz and a Hankel matrix:

$$\widetilde{T}_{c,a}(b) = [u_{i-j} + u_{i+j+1}].$$

A truly parallel argument (as in the previous sections) produces a very similar determinantal evaluation:

Theorem 6.1. *We have*

$$\det \widetilde{T}_{c,a}(b) = \prod_{i=1}^{c} \prod_{j=1}^{a} \frac{(b+i-j)(b+2i+j-1)}{(b+2i-j-1)(i+j-1)}.$$

To give some context, let $f(z)$ be a complex $L_1(\mathcal{C})$ function on the unit circle having Fourier coefficients

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta} d\theta, \qquad i = \sqrt{-1}.$$

Then, in the terminology of Widom [21], the *determinant of a Toeplitz matrix with symbol f* is given by

$$E_n(f) = \det[u_{j-i}].$$

Toeplitz determinants find homes in many topics, such as statistical physics and random matrix theory. Different types of matrix ensembles lend themselves to classes of Toeplitz plus Hankel matrices of the form

$$\det[u_{j-i} \pm u_{j+i}], \qquad \det[u_{j-i} \pm u_{j+i+1}], \qquad \text{etc.}$$

From a more mathematical vantage point, such determinants arise in the decomposition of the determinants $E_n(f)$. Namely, if the symbol f is an even function (in the sense $f(z) = f(\frac{1}{z})$ on the unit circle), then

$$\begin{aligned} E_{2n+1}(f) &= \frac{1}{2} \det[u_{j-i} - u_{j+i+2}]_0^{n-1} \times \det[u_{j-i} + u_{j+i}]_0^{n}, \\ E_{2n}(f) &= \quad \det[u_{j-i} - u_{j+i+1}]_0^{n-1} \times \det[u_{j-i} + u_{j+i}]_0^{n-1}. \end{aligned}$$

Andrews and Stanton ([1], see p. 274) involved a somewhat analogous splitting up of a certain determinant that arose in plane partitions although they did not view them as such.

7. BONUS: CT Identities and Plane Partitions

Borrowing notations from Section 5, the Toeplitz matrix $[u_{i-j}]$ that appeared as a component in our matrix $T_{c,a}(b)$ is in fact recognizable due to its relevance in the theory of plane partitions. Indeed, MacMahon [17] gave an elegant explicit formula for the number of plane partitions inside an $a \times b \times c$ box:

$$\prod_{i=0}^{a-1}\prod_{j=0}^{b-1}\prod_{k=0}^{c-1}\frac{i+j+k+2}{i+j+k+1}.$$

One may readily associate a determinantal formulation [16] to this enumeration

$$\det[u_{i-j}] = \det\left[\binom{a+b}{a-i+j}\right]_{i,j=0}^{c-1} = \prod_{i=0}^{a-1}\prod_{j=0}^{b-1}\prod_{k=0}^{c-1}\frac{i+j+k+2}{i+j+k+1}.$$

We have found what appears to be new (for us, at least) matrix whose determinant matches the above product formula and enjoys the expected symmetry.

Lemma 7.1. *It holds true that*

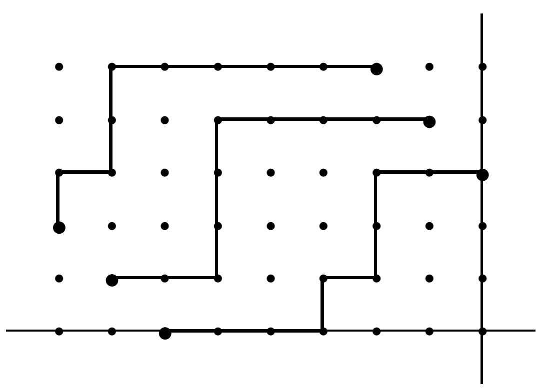
$$\det\left[\binom{i+j+a+b}{i+a}\right]_{i,j=0}^{c-1} = \prod_{i=0}^{a-1}\prod_{j=0}^{b-1}\prod_{k=0}^{c-1}\frac{i+j+k+2}{i+j+k+1}.$$

Proof. This is an immediate application of Dodgson's condensation [22]. □

Remark 2. The third author has previously employed the same approach towards furnishing a simple proof [23] of a determinant by MacMahon [17].

Proof. Here is a combinatorial proof for Lemma 7.1 given by C. Krattenthaler [15]. We thank him for allowing us to reproduce it.

Plane partitions in an $a \times b \times c$ box are in bijection with families $(P_0, P_1, \ldots, P_{c-1})$ of non-intersecting lattice paths, where P_i runs from $(-i-a, i)$ to $(-i, i+b)$, $i = 0, 1, \ldots, c-1$. This is explained in Section 3.3 of Bressoud's book [3]. An example for such a family of paths for $a = 6$ and $b = c = 3$ is displayed in the figure below.

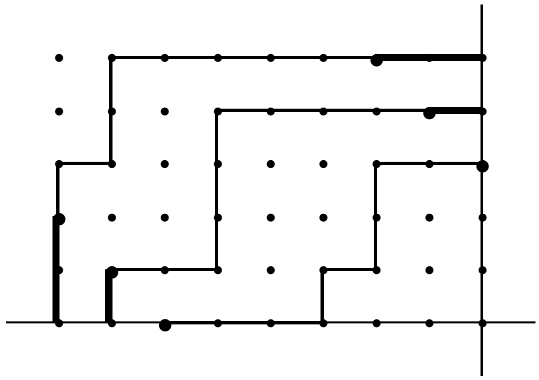


By the Lindström–Gessel–Viennot theorem [6], this leads to the determinant

$$\det_{0\le i,j\le c-1}\left(\binom{a+b}{a-i+j}\right)$$

for the enumeration of both objects being counted here.

We may prepend and append "forced" path pieces, see the thick path portions in the figure below.



In this manner, we obtain families $(Q_0, Q_1, \ldots, Q_{c-1})$ of non-intersecting lattice paths, where Q_i runs from $(-i-a, 0)$ to $(0, i+b)$, $i = 0, 1, \ldots, c-1$. The corresponding Lindström–Gessel–Viennot determinant [6] is

$$\det_{0\le i,j\le c-1}\left(\binom{i+j+a+b}{i+a}\right).$$

The proof is complete. □

In the context of Selberg's integrals, there are equivalently-stated constant term identities, and we prove (a special case of) one due to Morris [18] associated with the root system A_n.

Proposition 1. *We have*

$$CT_{\vec{t}} \prod_{i=0}^{c-1}(1+t_i)^a \left(1+t_i^{-1}\right)^b \prod_{i\neq j}^{0,c-1} \left(1-t_jt_i^{-1}\right)^m = \prod_{\ell=0}^{c-1} \frac{(a+b+\ell m)!\,((\ell+1)m)!}{(a+\ell m)!\,(b+\ell m)!\,m!}.$$

Proof. We are only proving the special case $m = 1$. Start with the determinant from Lemma 7.1 where the method of constant term extraction is brought to bear:

$$\begin{aligned}
\det\left[\binom{i+j+a+b}{i+a}\right] &= \det\left[CT_{t_i}\left(t_i^{i+a}\left(1+\frac{1}{t_i}\right)^{i+j+a+b}\right)\right]\\
&= \det\left[CT_{t_i}\left((1+t_i)^{i+a}\left(1+\frac{1}{t_i}\right)^{b}\left(1+\frac{1}{t_i}\right)^{j}\right)\right]\\
&= CT_{\vec{t}}\ \prod_{i=0}^{c-1}(1+t_i)^{i+a}\left(1+\frac{1}{t_i}\right)^{b}\cdot\det\left[\left(1+\frac{1}{t_i}\right)^{j}\right]\\
&= CT_{\vec{t}}\ \prod_{i=0}^{c-1}(1+t_i)^{i+a}\left(1+\frac{1}{t_i}\right)^{b}\cdot\prod_{i<j}^{0,c-1}(t_j^{-1}-t_i^{-1})\\
&= \frac{1}{c!}\,CT_{\vec{t}}\ \prod_{i=0}^{c-1}(1+t_i)^{a}\left(1+\frac{1}{t_i}\right)^{b}\cdot\prod_{i<j}^{0,c-1}(t_j-t_i)(t_j^{-1}-t_i^{-1})\\
&= \frac{1}{c!}\,CT_{\vec{t}}\ \prod_{i=0}^{c-1}(1+t_i)^{i+a}\left(1+\frac{1}{t_i}\right)^{b}\cdot\prod_{0\le i\neq j\le c-1}\left(1-\frac{t_j}{t_i}\right).
\end{aligned}$$

From Lemma 7.1, again, we gather that

$$CT_{\vec{t}} \prod_{i=0}^{c-1}(1+t_i)^{i+a}\left(1+\frac{1}{t_i}\right)^b \cdot \prod_{i\neq j}^{0,c-1}\left(1-\frac{t_j}{t_i}\right) = c! \cdot \prod_{i=0}^{a-1}\prod_{j=0}^{b-1}\prod_{k=0}^{c-1}\frac{i+j+k+2}{i+j+k+1}.$$

Further algebraic manipulation confirms the desired conclusion. □

We will strengthen the discussion by providing yet another new proof for a special case of a constant term identity due to Macdonald for the BC_n root system [18]. We need some preliminary work first. To this end, recall the *super Catalan numbers* [7] defined by

$$S_{i,j} := \frac{(2i)!\,(2j)!}{2\,i!\,j!\,(i+j)!}.$$

Lemma 7.2. *We have*

$$\det[S_{i+a,j+b}]_{i,j=1}^n = \frac{(-1)^{\binom{n}{2}}}{2^n\,n!}\prod_{i=1}^n \frac{(2a+2i)!\,(2b+2i)!\,i!}{(a+i)!\,(b+i)!\,(a+b+n+i)!}.$$

Proof. A direct application of Dodgson's Condensation settles the argument. □

Corollary 7.3. *Let $\vec{t} = (t_1,\dots,t_n)$. The following identity holds true*

$$\begin{aligned} CT_{\vec{t}} \prod_{i=1}^n (1-t_i)^\alpha \left(1-\frac{1}{t_i}\right)^\alpha (1-t_i^2)^\beta \left(1-\frac{1}{t_i^2}\right)^\beta \times \\ \prod_{i<j}\left(1-\frac{t_j}{t_i}\right)\left(1-\frac{t_i}{t_j}\right)(1-t_it_j)\left(1-\frac{1}{t_it_j}\right) \\ = \prod_{i=1}^n \frac{(2\beta+2i-2)!\,(2\alpha+2\beta+2i-2)!\,i!}{(\beta+i-1)!\,(\alpha+\beta+i-1)!\,(\alpha+2\beta+n+i-2)!}. \end{aligned}$$

Proof. Start with the observation that

$$S_{i+a,j+b} = (-1)^{i+a}\,CT_t \frac{(1-t)^{2i+2a}(1+t)^{2j+2b}}{2\,t^{i+a+j+b}}$$

and proceed as usual

$$\begin{aligned} \det[S_{i+a,j+b}]_1^n &= \det\left[(-1)^{i+a}\,CT_{t_i}\frac{(1-t_i)^{2i+2a}(1+t_i)^{2j+2b}}{2\,t_i^{i+a+j+b}}\right]_1^n \\ &= \gamma_n \det\left[CT_{t_i}\frac{(1-t_i)^{2i+2a}(1+t_i)^{2b+2}}{t_i^{i+a+b+1}}\left(2+t_i+\frac{1}{t_i}\right)^{j-1}\right] \\ &= \gamma_n\,CT_{\vec{t}}\prod_{i=1}^n \frac{(1-t_i)^{2i+2a}(1+t_i)^{2b+2}}{t_i^{i+a+b+1}}\prod_{i<j}\left(t_j+\frac{1}{t_j}-t_i-\frac{1}{t_i}\right) \\ &= \gamma_n\,CT_{\vec{t}}\prod_{i=1}^n \frac{(1-t_i)^{2a+2}(1+t_i)^{2b+2}(-2+t_i+\frac{1}{t_i})^{i-1}}{t_i^{a+b+2}}\prod_{i<j}(t_j-t_i)\left(1-\frac{1}{t_it_j}\right) \\ &= \frac{\gamma_n}{n!}\,CT_{\vec{t}}\prod_{i=1}^n \frac{(1-t_i)^{2a+2}(1+t_i)^{2b+2}}{t_i^{a+b+2}}\prod_{i<j}(t_j-t_i)^2\left(1-\frac{1}{t_it_j}\right)^2 \end{aligned}$$

$$= \frac{(-1)^{n(a+1)}\gamma_n}{n!} CT_{\vec{t}} \prod_{i=1}^{n} (1-t_i)^{a+1} \left(1-\frac{1}{t_i}\right)^{a+1} (1+t_i)^{b+1} \left(1+\frac{1}{t_i}\right)^{b+1}$$
$$\times \prod_{i<j} \left(1-\frac{t_j}{t_i}\right)\left(1-\frac{t_i}{t_j}\right)(1-t_it_j)\left(1-\frac{1}{t_it_j}\right);$$

where $\gamma_n := (-1)^{\binom{n+1}{2}}(-1)^{na}2^{-n}$. The rest follows from applying Lemma 7.3, from above, with $a = \alpha+\beta-1$ and $b = \beta-1$ and a few simplifications. □

We conclude this section and our paper by suggesting a natural inquiry for the inspired reader.

Question. *Is there a q-analogue of Conjecture 1?*

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