A WORDY PROOF OF A COMBINATORIAL LEMMA THAT AROSE IN OPERATOR THEORY

Doron ZEILBERGER[1]

Albert Einstein used to be amazed how effective mathematics is in science. Myself, I am amazed how effective combinatorics is in mathematics. Especially fruitful is the 'formal language approach' of *notre bon Maître*, Marco Schützenberger, pursued so vigorously and successfully by the *École Bordelaise* (see [B] and references thereof).

Recently, Baillon and Bruck[BB] proved a long-standing conjecture in operator theory, by reducing it to a hypergeometric identity that they proved using the package EKHAD[2] (accompanying the forthcoming book [PWZ],) They asked for a 'computer-free' proof. Such a proof was later given by Peter Paule[P]. However, the reduction of their result to the identity proved by EKHAD and Paule required considerable human effort on their part. In this note I will give a much shorter (and purely human!) proof by playing with (Motzkin[3]) words, dear to Viennot and his disciples.

Proposition([BB]): Let $c(m,n)$ be defined by $c(m+1,m)=0$, $(m \geq 0)$, $c(0,n)=1$, $(n \geq 0)$, and for $1 \leq m \leq n$, by the recurrence

$$c(m,n) = \mu c(m-1,n) + \mu c(m,n-1) + (1-2\mu)c(m-1,n-1) \quad ,$$

then the generating function of the 'diagonal',

$$\psi(t) := \sum_{m=0}^{\infty} c(m,m)t^m \quad ,$$

is given by

$$\psi(t) = 1 + \frac{\sqrt{1+\frac{4(1-\mu)\mu t}{1-t}} - 1}{2\mu} \quad .$$

Proof: Consider all words w in the alphabet $\{N,E,D\}$. For any word w let $y(w)$ be the number of Ns plus the number of Ds and let $x(w)$ be the number of Es plus the number of Ds. Let $\mathcal{M}$ be the set of words $w = l_1 l_2 \ldots l_r$ such for any prefix $w_i := l_1 l_2 \ldots l_i$ $(1 \leq i \leq r)$ one has $y(w_i) \geq x(w_i)$. Let $\mathcal{M}(m,n)$ be the subset of $\mathcal{M}$ for which $x(w)=m$ and $y(w)=n$.

Let the *weight* of the letters N and E be μ, and the weight of the letter D be $1-2\mu$. Introduce two weights on the words of $\mathcal{M}$. The normal weight, *weight*(w) is simply the product of the

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[1] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. `zeilberg@math.temple.edu`
`http://www.math.temple.edu/~zeilberg` `ftp://ftp.math.temple.edu/pub/zeilberg` .
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[2] Available from the http and ftp sites given in footnote 1. Once you downloaded it into your directory, get into Maple, and type: `read EKHAD;`, and follow the instructions given there

[3] It is interesting that often political leaders begat combinatorialists. Theodore Motzkin is the son of the Zionist leader Leo Motzkin (that has a town in Israel named after him, where, incidentally, I grew up). Other examples (of the sons) are Michel Mendès-France, Robin Wilson and Eri Jabotinsky.

weights of all the letters. For example $weight(NNDE) = \mu\mu(1-2\mu)\mu = \mu^3(1-2\mu)$. The modified weight, $weight'$ is as before, *except* that the initial string of Ns is given weight 1. For example $weight'(NNNDEDE) = 1^3 \cdot (1-2\mu)\mu(1-2\mu)\mu = (1-2\mu)^2\mu^2$. Let

$$C(m,n) := \sum_{w \in \mathcal{M}(m,n)} weight'(w) \quad .$$

Since, for $n \geq m > 0$, we have $\mathcal{M}(m,n)=\mathcal{M}(m-1,n)E \cup \mathcal{M}(m,n-1)N \cup \mathcal{M}(m-1,n-1)D$, and $\mathcal{M}(m+1,m)$ is the empty set, while $\mathcal{M}(0,n)$ consists of the single word N^n, taking $weight'$, we see that $c(m,n) = C(m,n)$.

Let $\overline{\mathcal{M}}$ be the set of all words w in $\mathcal{M}$ such that $x(w) = y(w)$, in other words (sic!), the set of all words in $\mathcal{M}$ with as many Ns as Es. Let $WEIGHT(w) := t^{x(w)} weight(w)$, and $WEIGHT'(w) := t^{x(w)} weight'(w)$. Since the 'syntax' of $\overline{\mathcal{M}}$ is given by:

$$\overline{\mathcal{M}} = \{empty \quad word\} \cup D\,\overline{\mathcal{M}} \cup N\,\overline{\mathcal{M}}\,E\,\overline{\mathcal{M}} \quad ,$$

taking $WEIGHT$ and $WEIGHT'$ respectively yields that $\phi(t) := \sum_{w \in \overline{\mathcal{M}}} WEIGHT(w)$ and $\psi(t) := \sum_{w \in \overline{\mathcal{M}}} WEIGHT'(w)$ satisfy the equations

$$\phi(t) = 1 + t(1-2\mu)\phi(t) + \mu^2 t \phi(t)^2 \quad ,$$

$$\psi(t) = 1 + t(1-2\mu)\phi(t) + \mu t \psi(t)\phi(t) \quad .$$

Now use the second equation to express $\phi(t)$ in terms of $\psi(t)$, plug into the first equation, clear denominators, and solve the resulting quadratic equation in $\psi(t)$, getting the expression for $\psi(t)$ claimed in the proposition. □

Remark: For their application, Baillon and Bruck were only interested in the asymptotics of $c(m,m)$, which is just as easily derived from the expression in the proposition above (which is Th. 5.5 in [BB]) as from their integral representation (Th. 6.1 there). However, one can easily prove (albeit using EKHAD's services) that these are equivalent, by typing (in EKHAD):
`AZpapd(sqrt(1+4*mu*(1-mu)*t/(1-t))/t**(m+2),t,m);`
and `AZpapd(sqrt(t/(1-t))*(1-4*mu*(1-mu)*t)**m,t,m);`

References

[BB] J.-B. Baillon and R.E. Bruck, *The rate of Asymptotic Regularity is* $O(1/\sqrt{n})$, preprint. Available from `bruck@mtha.usc.edu`.

[B] M. Bousquet-Mélou, "*q-Énumération de polyominos convexes*", Publications de LACIM, UQAM, Montréal, 1991.

[P] P. Paule, available from `peter.paule@risc.uni-linz.ac.at`.

[PWZ] M. Petkovsek, H.S. Wilf, and D.Zeilberger, "*A=B*", A.K.Peters, to appear.