The Experimental Mathematics of Voting

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Theorem 1 The weighted generating function of The Condorcet scenario $1 \to 2 \to 3 \to 1$ (with three candidates $1, 2, 3$ and according to the weight

$$x_{123}^{NumberOf123} \cdots x_{321}^{NumberOf321} t^{NumberOfVotes}$$

is

$$\frac{\left(1 + t^3 x_{123}\, x_{231}\, x_{312}\right) t^3 x_{123}\, x_{231} x_{312}}{\left(1 - t^2 x_{123}\, x_{321}\right)\left(1 - t^2 x_{213}\, x_{312}\right)\left(1 - t^2 x_{312}\, x_{123}\right)\left(1 - t^2 x_{132}\, x_{231}\right)\left(1 - t^2 x_{123}\, x_{231}\right)\left(1 - t^2 x_{231}\, x_{312}\right)} \quad .$$

Setting all the the x's to one, we get that the enumerating generating function for one of the Condorcet scenarios is

$$\frac{t^3 + t^6}{\left(1 - t^2\right)^6} \quad .$$

Hence the number of such scenarios with $2n - 1$ voters is the coefficient of t^{2n-1} in $\frac{t^3}{(1-t^2)^6}$, i.e. the coefficient of t^{2n-4} in $(1 - t^2)^{-6}$, i.e. the coefficient of t^{n-2} in $(1 - t^{-6}$, i.e. $(-1)^n \binom{-6}{n-2} = \binom{n+3}{5}$. The total number of compositions is the coefficient of t^{2n-1} in $(1 - t)^{-6}$, i.e. $\binom{-6}{2n-1} = \binom{2n+5}{5}$. Hence assuming that all compositions are *equally likely* (a very wrong assumption), the probability of one of the Condorcet scenarios, with $2n - 1$ voters is

$$\frac{(n-1)\, n\, (n+3)}{2\,(2\,n+1)\,(2\,n+3)\,(2\,n+5)} \quad .$$

So it tends to $\frac{1}{16}$ as $n \to 16$.

What if each voter chooses one of the six alternatives with the same probability (i.e. 1/6)?

Theorem 2:

Suppose three candidates, 1,2, and 3, are running for office, and there are $2n - 1$ voters, each of them choosing one of the 3!=6 possible ranking with the same probability (1/6). The votes are counted and it turns out that in the vote of

1 vs. 2, 1 won

2 vs. 3, 2 won

1 vs. 3, 3 won

In other words, there is a cycle $1 \to 2 \to 3 \to 1$.

Let a(n) be the probability of that happening times 6^{2n-1} (i.e. the numerator)

The integer sequence, $a(n)$, satisfies the following linear recurrence equation with polynomial coefficients

$$-1296\,\frac{n\,(2\,n+3)\,(2\,n+1)\,a\,(n)}{(n+1)\,(n+2)^2}+36\,\frac{(2\,n+3)\,\left(22\,n^2+33\,n+12\right)\,a\,(n+1)}{(n+1)\,(n+2)^2}$$
$$-4\,\frac{\left(19\,n^2+57\,n+45\right)\,a\,(n+2)}{(n+2)^2}+a\,(n+3)=0 \quad .$$

The asymptotic expression for $a(n)$ is $0.043869914022955 - 0.021101164\,n^{-1} + O(n^{-2})$.

By symmetry, this is also the probability of the scenario $1 \to 3 \to 2 \to 1$. So doubling, the asymptotic expression for the probability of a Condorcet scenario is

$$0.087739828045910 - 0.042202328\,n^{-1} \quad .$$

Corollary: The probability for a strict Condorcet scenario with three candidates and $2n-1$ voters tends as $n \to \infty$ to a constant, that is approximately equal to $0.087739828045910\ldots$, and that should be named the Condorcet constant.