THE UMBRAL TRANSFER-MATRIX METHOD: II. Counting Plane Partitions

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Abstract: We continue the 5-part saga on the Umbral Transfer-Matrix Method, based on Gian-Carlo Rota's seminal notion of the umbra. In this article we describe the Maple package PPar that automatically constructs Umbral Schemes for enumerating r-rowed plane partitions, for *specific* but *arbitrary* r, and that also automatically constructs *Evolution Umbra* for monotone triangles.

An Umbra for r-rowed plane partitions

Recall that an $r \times k$ plane-partition is a matrix of non-negative integers $(a_{i,j})$, $1 \leq i \leq r, 1 \leq j \leq k$, that is weakly decreasing along every row and every column. There is only one 'state' i.e. vertex-type, and it is parameterized by non-increasing tuples of non-negative integers: $(a_1, \dots, a_r)$, where $a_1 \geq a_2 \geq \dots \geq a_r \geq 0$. The Rota operator connecting one column to the next is the operator sending a generic monomial $x_1^{b_1} x_2^{b_2} \cdots x_r^{b_r}$ corresponding to the current leftmost column, to the sum of all monomials $(qx_1)^{a_1}(qx_2)^{a_2} \cdots (qx_r)^{a_r}$, where the sum ranges over all $a_1 \geq a_2 \geq \dots \geq a_r \geq 0$ for which $a_i \geq b_i$ for all i between 1 and r.

In order to do this automatically, we have to partition the summed set into 'chambers'. For any given i ($1 \leq i \leq r$), of course it is required that $a_i \geq b_i$, i.e. thinking of the $b_1, b_2, \dots, b_r$ as the heights of girls who stand in (weakly) decreasing order, and who each must find a (weakly) taller boy as her dancing partner, and the chosen boys must also be in (weakly) decreasing order, then a short girl is allowed to pick a boy that is even taller than her taller neighbor, in fact, the shortest girl may even pick a boy that is taller than the tallest girl. BUT, if she does that, then of course all the other boys must be taller than the tallest girl.

So for a fixed $(b_1, \dots, b_r)$, and a legal choice $(a_1, \dots, a_r)$, let's call the *profile* $f(i)$ the smallest j such that $a_i \geq b_j$. Of course $f(i) \leq i$, and $f(1) \leq f(2) \leq \dots \leq f(r)$. It is well known and easy to see that there are $C_r = \binom{2r}{r}/(r+1)$ such sequences, and they are sometimes called *Catalan sequences*. For example when $r = 2$ there are 2 Catalan sequences: 11 and 12, and when $r = 3$ there are 5: $111, 112, 113, 122, 123$.

Now for any specific profile, f, it is easy to sum $(qx_1)^{a_1}(qx_2)^{a_2} \cdots (qx_r)^{a_r}$, over all $(a_1, \dots, a_r)$ with that profile, since it only involves summing series of the form

$$P_m(A, B; z) := \sum_{A \leq i_1 \leq i_2 \leq \dots \leq i_m \leq B} z_1^{i_1} \dots z_m^{i_m} \quad ,$$

that Maple (and even you!) can evaluate in closed form (symbolically). Then the required sum is a product of a certain number of such expressions, where the number of terms in the prod-

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uct equals the cardinality of the range of f. For example the profile 1111 would only have $P_4(b_1, \infty; qx_1, qx_2, qx_3, qx_4)$ while the profile 1234 would be

$$P_1(b_1, \infty; qx_1) P_1(b_2, b_1 - 1; qx_2) P_1(b_3, b_2 - 1; qx_3) P_1(b_4, b_3 - 1; qx_4) \quad .$$

For details study carefully the source code of the Maple package `PPar`.

The next step is to find the pre-umbra by summing all these C_r expressions, and finally the *master* package `ROTA` converts the pre-umbra to a full-fledged Umbral Scheme. The function call is `UmbralScPP(x,q,t,r)`. Here x is the indexed variable, as above, q is the usual q that keeps track of the sum of the parts of the plane-partition, and t keeps track of the number of columns. Note that the zero vector is permitted, so the coeff. of t^k is really the generating function for plane partition with at most k columns and at most r rows, i.e. those whose 3D Ferrers diagram fit in an $r \times k \times \infty$ box, and in addition it is weighted by $x_1^{a_1} \dots x_r^{a_r}$, where $a_1, \dots, a_r$ is the leftmost column. For example, to get (essentially) the umbra derived by hand in [Z1] (for 2-rowed plane-partitions) type: `UmbralScPP(x,q,t,2)`.

To get the generating function for `n`-rowed, `r`-column plane partitions, using the variable `q`, type `GFPP(n,r,q);`, for specific `r` and `n`. For example try `GFPP(2,3,q);`. Of course, we already have MacMahon's box theorem, so we don't need `PPar`, but the point is to *illustrate* a general method that could be applied in many other situations, where closed form, or even efficient algorithms, are not known and may not exist. In fact `CheckPercy(n,r,q);` compares the answer to MacMahon's box theorem, and returns `true` if it agrees. If you want to keep track of the leftmost column, i.e. the weight is the previous one times $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $a_1, \dots, a_n$ are the entries of the leftmost row, then type: `GFPPx(n,x,r,q);` . Here `n` and `r` are *specific* integers, while x and q are symbols. For example, `GFPPx(3,x,2,q);`.

An Umbra for Monotone Triangles

Recall that a *monotone triangle* (a.k.a. *strict Gelfand pattern*) of size n is a triangular array of positive integers $(a_{i,j})$, $n \geq i \geq j \geq 1$, such that $a_{i,j} < a_{i,j+1}$ and $a_{i-1,j-1} \leq a_{i,j} \leq a_{i-1,j}$. The monotone triangles of size n whose bottom row is $1, 2, \dots, n$, i.e. for which $a_{n,j} = j$ for $j = 1, \dots, n$, are in simple bijection with $n \times n$ *alternating sign matrices.*

I believe that the Umbral Transfer Matrix method could be used to get yet another proof of the *Alternating Sign Matrix Conjecture* (see [B] for a gripping account). But it is not clear that this would be easier than the two already existing proofs. Since I have already paid my dues working on this problem, I will leave this as a challenge to the reader. Here I will only use it as an illustrative example, and describe the *automatic* generation of the generating function for *all* monotone triangles, but of a *specific* size. Then extracting the coefficient of $x_1 x_2^2 x_3^3 \dots x_n^n$ would give the n-th term in the famous ASM sequence.

Suppose that the bottom row is now $(b_1, \dots, b_k)$. The weight of such a monotone triangle is $x_1^{b_1} \cdots x_k^{b_k}$. Which tuples $(a_1, \dots, a_{k+1})$ may continue it to the $(k+1)$-th row? We need to sum

$x_1^{a_1} \cdots x_{k+1}^{a_{k+1}}$ over all legal $(k+1)$-tuples such that $a_1 < a_2 < \ldots < a_k < a_{k+1}$ and $a_1 \leq b_1 \leq a_2 \leq \ldots \leq b_k \leq a_{k+1}$. The summed-over set can be partitioned into 2^k subsets, according to *profile*. The profile of $(a_1, \ldots, a_{k+1})$ is the subset of $\{1, 2, \ldots k\}$ such that $b_i = a_{i+1}$. It is easy to explicitly (symbolically) sum the monomials corresponding to tuples belonging to any one profile. `PPar` does that for each of the 2^k profiles, getting expressions in $x_1, \ldots, x_{k+1}$ and $b_1, \ldots, b_k$. It adds them all up thereby getting the *preumbra* connecting a generic monomial $x_1^{b_1} \cdots x_k^{b_k}$ to the sum of all its legal extensions. Then procedure *ToUmbra* from the master package `ROTA` turns the pre-umbra into the Umbra T_k connecting monotone triangles of size k to those of size $k+1$. This is done with procedure `UmbraMT` of `PPar`. The function call is `UmbraMT(x,k);`,where x is the symbol for the indexed variable, and k is a *specific* integer. For example try `UmbraMT(x,3);`. To get the generating function for monotone triangles of size k (which is just $T_{k-1}T_{k-2} \ldots T_1(x_1)$), type `GFMT(x,k);`. To get the number of monotone triangles of size k with a prescribed bottom row (a list of increasing integers), type `MT(bottom_row);`. For example `MT([1,2,3,4,5]);` should yield the number of 5×5 alternating sign matrices (429), while `MT([1,3,5,7]);` gives the number of monotone triangles of size 4 whose bottom row is $1, 3, 5, 7$. Once again, note that this is *not* the most efficient way to crank out numbers, and the main justification of procedure `MT` is to test the validity of the program.

Other Symmetry Classes and Solid Partitions

The Same approach should enable one to construct *umbra*, or Umbral Schemes, for symmetric, totally symmetric, and other kinds of plane partitions. It should also be amenable to counting solid partitions, and more generally, generalized partitions that live on a poset, taking Stanley's classical theory of P-partitions ([S]) as a starting point, and building Umbral Schemes on it.

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