Tutorial on Applications of Expectation and Variance

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Example I: Two classes, one consisting of ONLY boys, that has 30 students, and another consisting of ONLY girls, that has 20 students, go to a field trip. Each class had one teacher.

One of the students is randomly selected, each student with equal probability. Let random variable X denote the number of children in that chosen student's class.

One of the teachers of these two classes is randomly selected, each with the same probability. Let the random variable Y denote the number of students of the chosen teacher's class.

(a) What is larger $E(X)$ or $E(Y)$? Explain why these quantities are different.

(b) Compute $Var(X)$ and $Var(Y)$

Solution to Example I:

(a)

There are $30 + 20 = 50$ students, 30 are boys and 20 are girls, and since each student is equally likely (with prob. $\frac{1}{50}$) to be picked:

$$P(Boy) = \frac{30}{50} = \frac{3}{5} \quad , \quad P(Girl) = \frac{20}{50} = \frac{2}{5} \quad .$$

Also, for the random variable X

$$X(Boy) = 30 \quad , \quad X(Girl) = 20 \quad .$$

Hence

$$E(X) = P(Boy) \cdot X(Boy) + P(Girl) \cdot X(Girl) = \frac{3}{5} \cdot 30 + \frac{2}{5} \cdot 20 = \frac{130}{5} = 26 \quad .$$

For the random variable Y, the values are analogous

$$Y(TeacherOfBoys) = 30 \quad , \quad Y(TeacherOfGirls) = 20 \quad ,$$

but the **probability distribution** is different:

$$P(TeacherOfBoys) = \frac{1}{2} \quad , \quad P(TeacherOfGrils) = \frac{1}{2} \quad .$$

Hence

$$E(Y) = P(TeacherOfBoys) \cdot Y(TeacherOfBoys) + P(TeacherOfGirls) \cdot Y(TeacherOfGirls)$$

$$= \frac{1}{2} \cdot 30 + \frac{1}{2} \cdot 20 = \frac{50}{2} = 25 \quad .$$

Answer to I(a): $E(X) = 26$ and $E(Y) = 25$. Hence $E(X)$ is larger. This is as it should be, since in this case the "coin" is **loaded** in favor of the boys, since there are more of them.

(b)

(b) regarding X

We already know from part (a) that $E(X)$, let's call it μ, equals 26.

So

$$Var(X) = E((X-\mu)^2) = \frac{3}{5} \cdot (30-26)^2 + \frac{2}{5} \cdot (20-26)^2 = \frac{3}{5} \cdot (4)^2 + \frac{2}{5} \cdot (-6)^2 = \frac{3}{5} \cdot 16 + \frac{2}{5} \cdot 36 = \frac{120}{5} = 24 \quad .$$

(b) regarding Y

We already know from part (a) that $E(Y)$, let's call it μ, equals 25.

So

$$Var(Y) = E((Y-\mu)^2) = \frac{1}{2} \cdot (30-25)^2 + \frac{1}{2} \cdot (20-25)^2 = \frac{1}{2} \cdot (5)^2 + \frac{1}{2} \cdot (-5)^2 = \frac{1}{2} \cdot 25 + \frac{1}{2} \cdot 25 = 25 \quad .$$

Answer to I(b): $Var(X) = 24$ and $Var(Y) = 25$.

Do Right Now: Problem 1 of Homework 5-PolySci 200A.

Important Properties of Expectation

Linearity of Expectation: For *any* random variables X, Y (with the same unerlying probability space of course)

[Warning: In Example I, this is not the case!], we have

$$E(X+Y) = E(X) + E(Y) \quad .$$

(In words: whenever you have to take the expectation of a sum, you can take the sum of the expectations, in other words, you can distribute)

Warning: $E(XY)$ is **NOT** $E(X)E(Y)$. It is only true if X and Y are *independent* random variables (they do not influence each other). In fact it is true, more generally, if they are *uncorellated.*

"Taking a Constant Out" Rule for Expectation

For any **constant** c

$$E(cX) = cE(X) \quad .$$

(In words: You can take **constants** out of the E symbol).

Note: This makes perfect sense. If everything gets, e.g., doubled, then the expectation is also doubled.

The expectation of a constant random variable is that constant!

If c is a constant, then

$$E(c) = c \quad .$$

Of course, if it is always the same (very boring!) the 'expectation' is *guaranteed* and is equal to that constant.

Important special cases:

$$E(1) = 1 \quad , \quad E(0) = 0 \quad .$$

Taking a Constant Out Rule for Variance

$$Var(cX) = c^2\, Var(X) \quad .$$

Warning: Note the c^2 on the right side, e.g. $Var(3X) = 9Var(X)$ **NOT** $3Var(X)$.

Adding a constant Rule Variance: If c is any constant, then

$$Var(c + X) = Var(X) \quad .$$

Comment: Adding a constant does not change the "variability", it gets cancelled out!

Example II:

If $E(X) = 3$ and $E(X^2) = 1$, and $E(X^3) = -1$. Find

(a) $E((X + 2)(X - 3))$

(b) $E(X(X + 2)(X - 3))$

Solution to Example II:

(a):

Using *highschool algebra*

$$(X+2)(X-3) = X^2 + 2X - 3X - 6 = X^2 - X - 6 \quad .$$

Applying the **expecation operation**, E, to both sides, we get

$$E(\,(X+2)(X-3)\,) = E(\,X^2 - X - 6\,) \quad .$$

Using the **linearity of expectation**, we get that the above equals

$$E(X^2) - E(X) - 6E(1) \quad .$$

From the data of the problem (see problem above), $E(X^2) = 1$, $E(X) = 3$, and of course (always!) $E(1) = 1$, so this equals

$$1 - 3 - 6 = -8 \quad .$$

Answer to II(a): -8.

(Note: For this part, $E(X^3)$ was not needed).

(b):

Using *highschool algebra*

$$X(X+2)(X-3) = X(X^2 + 2X - 3X - 6) = X(X^2 - X - 6) = X^3 - X^2 - 6X \quad .$$

Applying the **expecation operation**, E, to both sides, we get

$$E(\,X(X+2)(X-3)\,) = E(\,X^3 - X^2 - 6X\,) \quad .$$

Using the **linearity of expectation**, we get that the above equals

$$E(X^3) - E(X^2) - 6E(X) \quad .$$

From the data of the problem (see problem above),

$E(X^3) = -1$, $E(X^2) = 1$, $E(X) = 3$,

so this equals

$$(-1) - (1) - 6 \cdot (3) = -1 - 1 - 18 = -20 \quad .$$

Answer to II(b): -20.

Example III: If $E(X) = 3$ and $Var(X) = 2$, find

(a) $E(\,(2X+1)(X-2)\,)$

(b) $Var(-11+5X)$

Solution to Example III(a):

This is a **multi-step problem**. Before starting, we must find $E(X^2)$, and then we can proceed as in Example II.

By the famous formula

$$Var(X) = E(X^2) - E(X)^2 \quad ,$$

we get the equivalent statement (also worth memorizing)

$$E(X^2) = Var(X) + E(X)^2 \quad .$$

Hence, in this problem

$$E(X^2) = Var(X) + E(X)^2 = 2 + 3^2 = 2 + 9 = 11 \quad .$$

Now that we have *both* $E(X) = 3$, and $E(X^2) = 11$, we can proceed to answer III(a).

Using *highschool algebra*

$$(2X+1)(X-2) = 2X^2 + X - 4X - 2 = 2X^2 - 3X - 2 \quad .$$

Hence, applying the E operation to **both sides**, and using the *linearity of expectation*, and *taking out constants* rules

$$E(\,(2X+1)(X-2)\,) = E(\,2X^2 - 3X - 2\,) = 2E(X^2) - 3E(X) - 2E(1) =$$

$$= 2 \cdot 11 - 3 \cdot 3 - 2 \cdot 1 = 22 - 9 - 2 = 11 \quad .$$

Answer to III(a): 11.

Solution to III(b):

$$Var(-11+5X) = Var(5X) = 5^2 Var(X) = 25 \cdot 2 = 50 \quad .$$

Answer to III(b): 50.

Note: Part (b) is much easier and does **not** depend on part (a).

Do Right Now: Problem 2 of Homework 5-PolySci 200A.

Note on Problem 3 of Homework 5-PolySci 200A: This is a very wordy, multi-step problem, but you have all the technical know-how to tackle it. Please do your best to do it *all by yourself!*

Example IV:

Verify, directly, that

$$E(X+Y) = E(X) + E(Y) \quad ,$$

if X can only take the values 2 and 5, and Y can only take the values -1 and 4, and it is given that

$$P(X = 2 \quad AND \quad Y = -1) = \frac{1}{8} \quad ,$$

$$P(X = 2 \quad AND \quad Y = 4) = \frac{1}{4} \quad ,$$

$$P(X = 5 \quad AND \quad Y = -1) = \frac{1}{8} \quad ,$$

$$P(X = 5 \quad AND \quad Y = 4) = \frac{1}{2} \quad .$$

Solution to Example IV:

Let's figure out the **probability distribution** for the new random variable $X + Y$

$$P(X+Y = 1) = \frac{1}{8}$$

$$P(X+Y = 6) = \frac{1}{4} \quad ,$$

$$P(X+Y = 4) = \frac{1}{8} \quad ,$$

$$P(X+Y = 9) = \frac{1}{2} \quad .$$

Hence

$$E(X+Y) = P(X+Y=1)\cdot 1 + P(X+Y=6)\cdot 6 + P(X+Y=4)\cdot 4 + P(X+Y=9)\cdot 9$$

$$= \frac{1}{8}\cdot 1 + \frac{1}{4}\cdot 6 + \frac{1}{8}\cdot 4 + \frac{1}{2}\cdot 9 = \frac{53}{8} \quad .$$

So, by doing it directly, we got $E(X+Y) = \frac{53}{8}$.

Now let's find $E(X)$ and $E(Y)$. Using the above data:

$$P(X=2) = P(X = 2 \quad AND \quad Y = -1) + P(X = 2 \quad AND \quad Y = 4) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \quad ,$$

$$P(X=5) = P(X=5 \quad AND \quad Y=-1) + P(X=5 \quad AND \quad Y=4) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8} \quad .$$

Hence

$$E(X) = P(X=2)\cdot 2 + P(X=5)\cdot 5 = \frac{3}{8}\cdot 2 + \frac{5}{8}\cdot 5 = \frac{31}{8}$$

Similarly

$$P(Y=-1) = P(X=2 \quad AND \quad Y=-1) + P(X=5 \quad , AND \quad Y=-1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$P(Y=4) = P(X=2 \quad AND \quad Y=4) + P(X=5 \quad AND \quad Y=4) = \frac{2}{8} + \frac{1}{2} = \frac{3}{4}$$

Hence

$$E(Y) = P(Y=-1)\cdot(-1) + P(Y=4)\cdot 4 = \frac{1}{4}\cdot(-1) + \frac{3}{4}\cdot 4 = \frac{11}{4}$$

Hence

$$E(X) + E(Y) = \frac{31}{8} + \frac{11}{4} = \frac{53}{8} \quad .$$

The same as $E(X+Y)$ computed above!

YEA!: We verified the theorem $E(X+Y) = E(X) + E(Y)$ for this *particular* **special case**.

Warning: The example below is very advanced and abstract.

Example V:

Prove that

$$E(X+Y) = E(X) + E(Y)$$

for the special case where X and Y can each only take two different values.

Solution to Example V:

Let the two values that X can take be x_1 and x_2, and let the two values that Y can take be y_1 and y_2.

Also, let us denote the following:

$$P(X=x_1 \quad AND \quad Y=y_1) = p_{11} \quad ,$$

$$P(X=x_1 \quad AND \quad Y=y_2) = p_{12} \quad ,$$

$$P(X=x_2 \quad AND \quad Y=y_1) = p_{21} \quad ,$$

$$P(X=x_2 \quad AND \quad Y=y_2) = p_{22} \quad ,$$

where of course, $p_{11}+p_{12}+p21+p_{22}=1$ (they must add-up to 1, since this covers **all** possibilities), and they are each between 0 and 1 (inclusive), as probabilities should be.

Let's figure out the **probability distribution** for the new random variable $X+Y$

$$P(X+Y=x_1+y_1)=p_{11}$$

$$P(X+Y=x_1+y_2)=p_{12} \quad ,$$

$$P(X+Y=x_2+y_1)=p_{21} \quad ,$$

$$P(X+Y=x_2+y_2)=p_{22} \quad .$$

Hence

$$E(X+Y)=P(X+Y=x_1+y_1)\cdot(x_1+y_1)+P(X+Y=x_1+y_2)\cdot(x_1+y_2)$$
$$+P(X+Y=x_2+y_1)\cdot(x_2+y_1)+P(X+Y=x_2+y_2)\cdot(x_2+y_2)$$
$$=p_{11}(x_1+y_1)+p_{12}(x_1+y_1)+p_{21}(x_2+y_1)+p_{22}(x_2+y_2) \quad .$$

Now let's find $E(X)$ and $E(Y)$.

Using the above data:

$$P(X=x_1)=P(X=x_1 \quad AND \quad Y=y_1)+P(X=x_1 \quad AND \quad Y=y_2)=p_{11}+p_{12} \quad ,$$

$$P(X=x_2)=P(X=x_2 \quad AND \quad Y=y_1)+P(X=x_2 \quad AND \quad Y=y_2)=p_{21}+p_{22} \quad .$$

Hence

$$E(X)=P(X=x_1)\cdot x_1+P(X=x_2)\cdot x_2=(p_{11}+p_{12})x_1+(p_{21}+p_{22})x_2 \quad .$$

Similarly,

$$P(Y=y_1)=P(X=x_1 \quad AND \quad Y=y_1)+P(X=x_2 \quad AND \quad Y=y_1)=p_{11}+p_{21} \quad ,$$

$$P(Y=y_2)=P(X=x_1 \quad AND \quad Y=y_2)+P(X=x_2 \quad AND \quad Y=y_2)=p_{12}+p_{22} \quad .$$

Hence

$$E(Y)=P(Y=y_1)\cdot y_1+P(Y=y_2)\cdot y_2=(p_{11}+p_{21})y_1+(p_{12}+p_{22})y_2 \quad .$$

Adding up, we get:

$$E(X)+E(Y)=(p_{11}+p_{12})x_1+(p_{21}+p_{22})x_2+(p_{11}+p_{21})y_1+(p_{12}+p_{22})y_2 \quad ,$$

and this the same as $E(X+Y)$ computed above! (check the algebra!) QED.

Note: This is a more general version of Problem 4 in Homework 5. There it is assumed that the two random variables are *independent*.

There the inputs are

$P(X=x_1)$ (and $P(X=x_2)$, that is necessarily $1-P(X=x_1)$, since there are only two possibilities for the values of X),

and

$P(Y=y_1)$ (and $P(Y=y_2)$, that is necessarily $1-P(Y=y_1)$, since there are only two possibilities for the values of Y).

Calling $P(X=x_1)=\alpha$ and $P(Y=y_1)=\beta$, to do Probelm 4 of Homework 5, you do the above, but with the simplifying values

$$p_{11}=\alpha\beta \quad , \quad p_{12}=\alpha(1-\beta) \quad ,$$

$$p_{21}=(1-\alpha)\beta \quad , \quad p_{22}=(1-\alpha)(1-\beta) \quad , \quad .$$

Example VI: Let X be a random variable with expected value μ and variance σ^2. Find the expected values of the following functions of X

$Y=\frac{3X-5\mu}{7\sigma}$

Solution to Example VI:

$$E(Y)=E(\frac{3X-5\mu}{7\sigma})=\frac{3E(X)-5\mu}{7\sigma}=\frac{3\mu-5\mu}{7\sigma}=-\frac{2\mu}{7\sigma}$$

Do right now: Problem 5 of Homework 5.

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