Automatic Generation of Generating Functions for Chromatic Polynomials for Grid Graphs (and more general creatures) of Fixed (but arbitrary!) Width

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Very Important: This article comments on the Maple package `http://www.math.rutgers.edu/~zeilberg/tokhniot/KamaTzviot`. Some sample input and output can be gotten from the "front" of this article:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/tzeva.html` .

One of the *many* things done in, or inspired by, the famous INRIA **ALGORITHMS PROJECT** (http://algo.inria.fr/), the brain child of our beloved guru **Philippe Flajolet**, was to devise efficient algorithms for computing *chromatic polynomials* for *grid graphs*, namely the Cartesian product $\mathcal{P}_n \times \mathcal{P}_m$ where $\mathcal{P}_n$ is the path of length n. It is fairly easy to see that for a fixed m, the generating function

$$F_m(z,c) = \sum_{n=0}^{\infty} P_{\mathcal{P}_n \times \mathcal{P}_m}(c) z^n$$

is a *rational function* of **both** c and z. The Maple package `KamaTzviot` automatically computes these rational function for *any* inputted numeric m. (See procedure `GFk(m,t,c)` of `KamaTzviot`).

In fact we do something much more general. For *any* graph G, `KamaTzviot` can (explicitly!) compute

$$F_G(z,c) = \sum_{n=0}^{\infty} P_{\mathcal{P}_n \times G}(c) z^n \quad .$$

(See procedure `GFG(G,t,c)` .)

In fact we do something *even* more general! For *any* graph G, on m vertices, and for *any* bipartite (m,m) graph C, let $M_n(G,C)$ be the graph on mn vertices where the edges among

$$1+im\,,\ 2+im\,,\ \dots,\ m+im$$

mimic the graph G (for $i = 0, \dots, n-1$), and in addition the edges between

$$1+im\,,\,2+im\,,\,...\,,\,m+im$$

and

$$1+(i+1)m\,,\,2+(i+1)m\,,\,...\,,\,m+(i+1)m$$

$(0 \leq i < n-1)$ mimic the edges of C, given as a set of (up to m^2) ordered pairs $\{[\alpha,\beta]\}$. $[\alpha,\beta] \in C$ means that there is an edge between vertex $\alpha+im$ and vertex $\beta+(i+1)m$ for $0 \leq i < n-1$. Note that when C is the monogamy bipartite graph $\{[1,1],\ldots,[m,m]\}$, where Mr i is connected to Mrs i (but no cheating!), then $M_n(G,C)$ reduces to the Cartesian product $G \times \mathcal{P}_n$.

`KamaTzviot` can (explicitly!) compute the rational function (of z and c):

$$F_{G,C}(z,c) = \sum_{n=0} P_{M_n(G,C)}(c) z^n \quad .$$

See procedure `GFGG(G,C,t,c)`.

The Method

Of course we use the *transfer matrix method*, but in a *symbolic* context. The graph G has m vertices, so it can be legally colored by at most m colors. Let's label the vertices of G, once and for all, by the integers $\{1,\ldots,m\}$, and visualize them from left to right. For each (legal) vertex-coloring of G we can associate a *canonical* form, by renaming the color of vertex 1, color 1, then the color of the smallest vertex of G that is not colored by color 1, color 2, and then the color of the smallest vertex colored by neither of these colors, color 3, etc.

For example, If the coloring is 351132 then its canonical form is 123314.

Our set of "states" consists of all possible canonical colorings of G. Of course, since G has finitely many vertices, this set is a finite set. (In `KamaTzviot` this is done by procedure `CC(G)`, `CC` stands for Canonical Colorings). Note that if G has no edges, then $CC(G)$ is in bijection with the set of set-partitions of $\{1,\ldots,m\}$, so an upper bound for the number of states is the Bell number B_m.

Suppose that the "bottom" G in $M_n(G,C)$ is in a certain state S, and we want to add another "layer" (a copy of C and G) to form $M_{n+1}(G,C)$, and we want the state of the new bottom to be T. In how many ways can we color the new m vertices (namely vertices $mn+1,\ldots,mn+m$) with c colors, so that the new coloring is still a legal vertex-coloring? If T is the list $[j_1,j_2,\ldots,j_m]$ (of course $j_1=1$), let's call the *actual* colors colored by these vertices $i_{j_1},\ldots,i_{j_m}$ respectively, and suppose that there are k different colors in that last layer, i.e.

$$\{j_1,\ldots,j_m\} = \{1,\ldots,k\} \quad ,$$

so that

$$\{i_{j_1},\ldots,i_{j_m}\} = \{i_1,\ldots,i_k\} \quad .$$

By the definition of "state" any such coloring would not introduce an edge *within* this last layer connecting vertices with the same color, but we have also to worry about the edges of the last installment of C. If the state S is

$$[a_1, \ldots, a_m] \quad ,$$

and it has s different colors, i,e,

$$\{a_1, \ldots, a_m\} = \{1, \ldots, s\} \quad ,$$

then for any edge $[\alpha, \beta] \in C$, we know, by the construction of $M_n(G, C)$, that there is an edge between $mn + \alpha$ and $m(n+1) + \beta$. This entails that i_{j_β} can **not** be a_α. So for each and every $i_1, \ldots, i_k$ there is a set of "forbidden colors", out of the s colors of layer n. Of course each of $i_1, \ldots, i_k$ can also be colored in one of the $c - s$ colors that are not used in layer n. Converting the "negative" conditions into "positive" ones, we get the set of permissible colors for each $i_1, \ldots, i_k$, including what we called "option 0" (in procedure `TS1S2G` of our Maple package) that denotes choosing one of the remaining $c - s$ colors. Taking the Cartesian product of the option-sets for each of $i_1, \ldots, i_k$, we get atomic events where some of the members of $\{i_1, \ldots, i_k\}$ are committed to be one of the s colors of layer n, and the rest are *different* colors from those $c - s$ colors. If there are γ such "0"s, of course the number of ways of doing it is the polynomial of degree γ in c, $\gamma!\binom{c-s}{\gamma}$. Adding the number of possibilities of all the atomic options would give the matrix-entry connecting state S to state T.

We (or rather the first-named author) does it for each pair of states S and T, building up the *transfer matrix* **completely automatically** (in other words, it does the "combinatorial research" all by itself!). Once we have the transfer matrix, Ekhad sets up the obvious set of equations for the generating functions for colorings ending at any given state, solves, (symbolically and automatically!) the resulting set of linear equations (with coefficients that are poynomials of z and c), and then adds them up to get the desired generating function.

Let us conclude with a simple example of the grid graphs $P_3 \times P_n$. Here

$$Edges(G) = \{\{1, 2\}, \{2, 3\}\} \quad ,$$

$$C = \{[1, 1], [2, 2], [3, 3]\} \quad .$$

There are two states: 121 and 123. Let's compute the matrix entry connecting state 123 to 121.

The coloring of the bottom layer is $[i_1, i_2, i_1]$ for *different* colors i_1 and i_2 (chosen between 1 and c).

Because of the edge $[1, 1]$ of C we have the restriction $i_1 \neq 1$.

Because of the edge $[2, 2]$, we have the restriction $i_2 \neq 2$.

Because of the edge $[3, 3]$, we have the restriction $i_1 \neq 3$.

So i_1 may **not** be colored with color 1 and color 3, while i_2 may **not** be colored with color 2. Going to the **positive** rephrasing:

$i_1 = 2$ OR $3 \leq i_1 \leq c$

AND

$i_2 = 1$ OR $i_2 = 3$ OR $3 \leq i_2 \leq c$.

Taking the "product" we have the six "atomic" events:

$i_1 = 2$ AND $i_2 = 1$ (1 possibility)

OR

$i_1 = 2$ AND $i_2 = 3$ (1 possibility)

OR

$i_1 = 2$ AND $3 \leq i_2 \leq c$ ($1!\binom{c-3}{1} = c-3$ possibilities)

OR

$3 \leq i_1 \leq c$ AND $i_2 = 1$ ($1!\binom{c-3}{1} = c-3$ possibilities)

OR

$3 \leq i_1 \leq c$ AND $i_2 = 3$ ($1!\binom{c-3}{1} = c-3$ possibilities)

OR

$3 \leq i_1 \leq c$ AND $3 \leq i_2 \leq c$ ($2!\binom{c-3}{2} = (c-3)(c-4)$ possibilities) .

Adding these up gives the matrix entry:

$$M[123,121] = 2 \cdot 1 + 3(c-3) + (c-3)(c-4) = c^2 - 4c + 5 \quad .$$

We leave to our *human* readers, as an instructive exercise to test their comprehension of our method, to verify that

$M[121,121] = c^2 - 3c + 3$, $M[121,123] = c^3 - 6c^2 + 13c - 10$, $M[123,123] = c^3 - 6c^2 + 14c - 13$.

Of course, `KamaTzviot` can do so much **more**.

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