Human and automated approaches for finite trigonometric sums

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Abstract

We show that identities involving trigonometric sums recently proved by Harshitha, Vasuki and Yathirajsharma, using Ramanujan's theory of theta functions, were either already in the literature or can be proved easily by adapting results that can be found in the literature. Also we prove two conjectures given in that paper. After mentioning many other works dealing with identities for various trigonometric sums, we end this paper by describing an automated approach for proving such trigonometric identities.

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Identities between quantities that are not clearly and immediately related are always fascinating. In particular when their proofs use subtle or complicated arguments (think of the Basel Problem, and the solution by Euler, $\sum 1/n^2 = \pi^2/6$). But it also happens that, after a "complicated" proof is found, people try to find a "direct" or an "elementary" proof: in some cases this is not (not yet?) possible (think of the Fermat-Wiles theorem). Trigonometric identities, *a priori* unexpected, can also be found in the literature. We will just cite two such identities, one due to Gauss (see, e.g., [27], where some signs seem to be misprinted):

$$\cos\left(\frac{\pi}{17}\right) = \frac{1-\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{34+2\sqrt{17}}}}{16}$$

the other is a consequence of Ramanujan theta function identities (take the first two equalities of Lemma 3.8 in [7], subtract the second one from the square of the first one, or see [4, Corollary 32.1, p.

184]; also see Eq. (1.31) of Corollary 7 in [38], and Eq. (1.11) in [32]):

$$\frac{\sin^2\left(\frac{3\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)} - \frac{\sin^2\left(\frac{2\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)} + \frac{\sin^2\left(\frac{\pi}{7}\right)}{\sin\left(\frac{3\pi}{7}\right)} = 0.$$

Much more modestly we stumbled upon the recent paper [32] published in the Ramanujan journal, where the authors give an interesting way to obtain a closed form for several trigonometric sums by using Ramanujan's theory of theta functions. In particular Theorem 1.1 in that paper gives six formulas about which the authors write that these six identities *seem to be new*. Of course it is very interesting to obtain these equalities through theta functions, but we were interested to search whether these formulas previously appeared in the literature. Having found a vast set of papers on the subject of identities involving finite trigonometric sums, we were lucky enough to discover either references in which some of these identities had already been proved, or known results from which the other identities can be easily deduced. Here we describe or adapt these results. We also confirm two conjectures given by the authors of [32] at the end of their paper.

Finally we show how to give automated proofs for trigonometric identities of this kind.

1 The six identities of Harshitha, Vasuki and Yathirajsharma

First we recall the six trigonometric identities given in [32, Theorem 1.1].

1. *If k is an odd natural number, then*

$$\sum_{j=1}^{\frac{k-1}{2}} (-1)^{j-1} \sin\left(\frac{(2j-1)\pi}{2k}\right) = \frac{(-1)^{\frac{k-3}{2}}}{2} \quad \textit{(here one should suppose that } k \geq 3\textit{)} \tag{1}$$

and

$$\sum_{j=1}^{\frac{k-1}{2}} (-1)^{j-1} \csc\left(\frac{(2j-1)\pi}{2k}\right) = \frac{k+(-1)^{\frac{k+1}{2}}}{2}. \tag{2}$$

2. *If n is an odd natural number and $j = 2p$ is an even positive integer such that* $\gcd(j,n) = 1$, *then*

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \frac{n^2-1}{3}. \tag{3}$$

3. *If n is an even number and $j \equiv 2 \pmod 4$, such that* $\gcd(\frac{j}{2}, \frac{n}{2}) = 1$, *then*

$$\sum_{k=0}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \frac{n^2}{4}. \tag{4}$$

4. *If n is an even number and $j \equiv 2 \pmod 4$, such that* $\gcd(\frac{j}{2}, \frac{n}{2}) = 1$, *then*

$$\sum_{k=0}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)k\pi}{n}\right)\sin\left(\frac{(j-1)k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)\sin^2\left(\frac{jk\pi}{n}\right)} = \frac{n^2-4}{12}. \tag{5}$$

5. *If n is an odd natural number and j is a positive integer such that* $\gcd(j,n)=1$, *then*

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{n}\right)\sin^2\left(\frac{j(2k+1)\pi}{n}\right)} = 0. \tag{6}$$

2 Some related trigonometric identities found in the literature

Here we give three propositions that state results we found in the literature.

Proposition 1 ([43], last displayed formula)

$$\sum_{j=1}^{n}(-1)^{j+1}\sin\left(\frac{(2j-1)\pi}{4n+2}\right) = \frac{(-1)^{n+1}}{2}\cdot$$

Remark 1 Several sums of the same kind appear in the literature, where sin can be replaced with cos or a power of cos or sin, for example a companion formula of the formula above is given in [43], namely, for $n \geq 1$,

$$\sum_{j=1}^{n}(-1)^{j+1}\cos\left(\frac{j\pi}{2n+1}\right) = \frac{1}{2}\cdot$$

Also the relations

$$\sum_{k=1}^{n}(-1)^{k+1}\cos^2\left(\frac{k\pi}{2n+2}\right) = \frac{1}{2} \quad \text{and} \quad \sum_{k=1}^{n}(-1)^{k+1}\sin^2\left(\frac{k\pi}{2n+2}\right) = \frac{(-1)^{n+1}}{2}$$

can be found in [52]. The same formulas can be found in [2, Theorem 3.4, p. 218].

Note that there is a straightfoward proof of the summation in Proposition 1 as well as of the three summations in this remark, by using the relation (see, e.g., [11, Example 7, p. 405], or use the complex exponential form of cos and sin)

$$(\#) \qquad \cos(\alpha)+\cos(\alpha+\beta)+\cdots+\cos(\alpha+(K-1)\beta) = \frac{\sin\left(\frac{1}{2}\beta K\right)}{\sin\left(\frac{1}{2}\beta\right)}\cos\left(\alpha+\frac{1}{2}\beta(K-1)\right)$$

and replacing the possible occurrences of $\cos^2(x)$ and $\sin^2 x$ with $(1\pm\cos 2x)/2$.

Proposition 2 ([33], Relation 3.16 on p. 817)

$$\sum_{j=0}^{n-1}(-1)^j\csc\left(\frac{(2j+1)\pi}{4n+2}\right) = n+\frac{1-(-1)^n}{2}\cdot$$

Now we give the last proposition of this section that we discovered in a 1908 book by Bromwich.

Proposition 3 ([8], p. 183; also see p. 187)

$$\begin{cases} \text{(I)} & \displaystyle\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} & = & \dfrac{n^2-1}{6} & \textit{if } n \textit{ is odd;} \\ \text{(II)} & \displaystyle\sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} & = & \dfrac{n^2-4}{6} & \textit{if } n \textit{ is even;} \\ \text{(III)} & \displaystyle\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} & = & \dfrac{n^2-1}{2} & \textit{if } n \textit{ is odd;} \\ \text{(IV)} & \displaystyle\sum_{k=0}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} & = & \dfrac{n^2}{2} & \textit{if } n \textit{ is even.} \end{cases}$$

Remark 2 One can find in [24, Identity (23)] the identity

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} = \frac{n^2-1}{3}.$$

The case where n is odd is an exercise in Bromwich's book ([8, Ex. 4, p. 188]). Also see [31] where it is stated that such identities were conjectured while computing low-temperature series for a Z_n-symmetric Hamiltonian in statistical mechanics [35]; (also see [34] and the references therein for much more on this and similar identities). Actually this identity can be easily deduced from (I) and (II) in Proposition 3 above (cut the sum into $\sum_{k=1}^{(n-1)/2} + \sum_{(n+1)/2}^{n-1}$ if n is odd, and into $\sum_{k=1}^{n/2} + \sum_{(n/2)+1}^{n-1}$ if n is even, and make the change of index $\ell = n - k$ in each of the second sums).

Remark 3 One can find in [9] (also see the other references given there) the identity

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} = n^2.$$

Note that this equality is given in [32, Remark 2] under the slightly disguised form

$$\sum_{k=1}^{n} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} = n^2$$

as a corollary of the identity

$$\sum_{k=1}^{n-1} \cot^2\left(\frac{k\pi}{n}\right) = \frac{(n-1)(n-2)}{3} \quad \text{for any } n > 0.$$

The authors of [32] cite a similar derivation in [5, Corollary 2.4]. It is interesting to note that the "disguised" equality above is already given (actually only for n even) under a slightly different disguise by Riesz in 1914 in [46, 47] where it is used to prove a theorem of Bernstein (about Riesz' results also see [60, Volume II, p. 10, Eq. 3.11] and [12]). Actually the identity at the beginning of this Remark 3 can be easily deduced from (III) and (IV) in Proposition 3 above, as was done in Remark 2.

Remark 4 Note that the identities (I, II, III, IV) in Proposition 3 are, up to notation, respectively the identities (3.40, 3.28, 3.14, 3.4) in the paper [33], where they are also generalized.

We end this section with an easy lemma partly deduced from what precedes.

Lemma 1 *The following equalities hold.*

(i) Let n be an odd integer and p an integer such that $\gcd(p,n)=1$. *Then*

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{\ell=1}^{n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \frac{1}{2}\sum_{r=1}^{n-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \frac{n^2-1}{6}\cdot$$

(ii) Let n be an even integer and p an odd integer, such that $\gcd(p,n/2)=1$. *Then*

$$\sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{\ell=1}^{n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{r=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \frac{n^2}{4}\cdot$$

Proof. (i) For n odd and $\gcd(p,n)=1$ we have

$$\begin{aligned}
\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{\substack{1\le \ell\le n-2\\ \ell \text{ odd}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)}\\
&= \sum_{1\le \ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ even}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)}\\
&= \sum_{1\le \ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.
\end{aligned}$$

But

$$\sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \sum_{\frac{n+1}{2}\le s\le n-1} \frac{1}{\sin^2\left(\frac{2ps\pi}{n}\right)} \quad \text{(change of index } s=n-r).$$

Hence

$$\sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \frac{1}{2}\left(\sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} + \sum_{\frac{n+1}{2}\le s\le n-1} \frac{1}{\sin^2\left(\frac{2ps\pi}{n}\right)}\right) = \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.$$

Thus, finally

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{1\le \ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.$$

Now we claim that the first sum on the right hand side only depends on the value of $p\ell$ modulo n and that $p\ell$ takes exactly once all the values modulo n except the value 0 since $\gcd(p,n) = 1$, thus this sum is equal to the same sum where p is replaced with 1. Similarly the second sum only depends on the value of $2pr$ modulo n and $2pr$ takes exactly once all the values modulo n except the value 0 since $\gcd(2p,n) = 1$, thus the sum is equal to the same sum where $2p$ is replaced with 1. Thus

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{\ell\pi}{n}\right)} - \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{r\pi}{n}\right)} = \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{r\pi}{n}\right)}.$$

This last sum is equal to $\frac{n^2-1}{6}$ from Remark 2, where it was indicated that the first identity there is a consequence of Proposition 3 (I).

(*ii*) For n even, p odd and $\gcd(p, n/2) = 1$ we have

$$\begin{aligned}
\sum_{k=0}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{\substack{1\le\ell\le n-1\\ \ell \text{ odd}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)}\\
&= \sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{\substack{1\le\ell\le n-1\\ \ell \text{ even}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)}\\
&= \sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{1\le r\le \frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.
\end{aligned}$$

As seen above the first sum on the right hand side does not depend on p. It is equal to $\frac{n^2-1}{3}$. Similarly, defining $m := n/2$, the second sum can be written

$$\sum_{1\le r\le m-1} \frac{1}{\sin^2\left(\frac{pr\pi}{m}\right)}$$

where $\gcd(p,m) = 1$. It is thus equal to $\frac{m^2-1}{3}$. Hence, finally

$$\sum_{1\le k\le \frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \frac{n^2}{4}.$$

Remark 5 In the course of the proofs above, we proved, for $\gcd(p,n) = 1$, the identity

$$\sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} = \frac{n^2-1}{3}$$

(compare with Remark 2).

3 Identities (1) to (6) revisited

We begin this section with a straightforward trigonometric formula, namely

$$(\#) \qquad \frac{\sin x \sin y}{\sin^2(\frac{x+y}{2})\sin^2(\frac{x-y}{2})} = \frac{1}{\sin^2(\frac{x-y}{2})} - \frac{1}{\sin^2(\frac{x+y}{2})}$$

(write: $2\sin x\sin y = \cos(x-y) - \cos(x+y)$ and use $\cos z = 1 - 2\sin^2(z/2)$).

Now we revisit Identities (1) to (6), to show that they were either already in the literature or easy consequences of identities in the literature.

- Identity (1)

  See Proposition 1.

- Identity (2)

  See Proposition 2.

- Identity (3)

  We have for n odd, $j = 2p$ and $\gcd(j,n) = 1$, successively using (#), Proposition 3 (III), and Lemma 1 (ii),

$$\begin{aligned}\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} - \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} \\ &= \frac{n^2-1}{2} - \frac{n^2-1}{6} = \frac{n^2-1}{3}.\end{aligned}$$

- Identity (4)

  We have for n even, $j = 2p$, $j \equiv 2 \pmod 4$, and $\gcd(\frac{j}{2}, \frac{n}{2}) = 1$ successively using (#), Proposition 3 (IV), and Lemma 1 (i)

$$\begin{aligned}\sum_{k=1}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} - \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} \\ &= \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}.\end{aligned}$$

- Identity (5)

  We have for n even, $j = 2p$, $j \equiv 2 \pmod 4$, and $\gcd(j/2, n/2) = 1$, successively using (#), Proposition 3 (II), and the identity in Remark 5,

$$\begin{aligned}\sum_{k=1}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)k\pi}{n}\right)\sin\left(\frac{(j-1)k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)\sin^2\left(\frac{jk\pi}{n}\right)} &= \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} - \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{jk\pi}{n}\right)} \\ &= \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} - \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{(j/2)k\pi}{n/2}\right)} \\ &= \frac{n^2-4}{6} - \frac{(\frac{n}{2})^2-1}{3} = \frac{n^2-4}{12}.\end{aligned}$$

- Identity (6)

  We have for n odd and j such that $\gcd(j,n) = 1$, successively using (#), and Lemma 1(i),

$$\begin{aligned}\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{n}\right)\sin^2\left(\frac{j(2k+1)\pi}{n}\right)} &= \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{n}\right)} - \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{j(2k+1)\pi}{n}\right)} \\ &= 0.\end{aligned}$$

4 Confirming two conjectures of Harshitha, Vasuki, Yathirajsharma

At the end of their paper [32], the authors give two conjectures about which they wrote: *We feel that these two can also be tackled with Ramanujan's theory although we were unable to do so.* First we recall these conjectures.

Conjecture [32] *If k is any positive integer, then*

$$\lim_{k\to\infty}\left\{\sum_{j=0}^{2k-1}(-1)^j \sin^2\left(\frac{(2j+1)\pi}{8k+2}\right)\right\} = -\frac{1}{2}$$

and

$$\lim_{k\to\infty}\left\{\sum_{j=0}^{2k}(-1)^j \sin^2\left(\frac{(2j+1)\pi}{8k+6}\right)\right\} = \frac{1}{2}\cdot$$

Here we give a simple proposition (but without Ramanujan's theory of theta functions) which easily implies both conjectures above.

Proposition 4 *The following equalities hold.*

$$\sum_{j=0}^{2k-1}(-1)^j \sin^2\left(\frac{(2j+1)\pi}{8k+2}\right) = \frac{-\sin^2\left(\dfrac{2k\pi}{4k+1}\right)}{2\cos\left(\dfrac{\pi}{4k+1}\right)}$$

and

$$\sum_{j=0}^{2k}(-1)^j \sin^2\left(\frac{(2j+1)\pi}{8k+6}\right) = \frac{1}{2} - \frac{\cos^2\left(\dfrac{(2k+1)\pi}{4k+3}\right)}{2\cos\left(\dfrac{\pi}{4k+3}\right)}\cdot$$

As a consequence the conjectures above hold.

Proof. The proof is left to the reader who can use the relations $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $(-1)^t\cos y = \cos(t\pi + y)$, and, as previously, Identity (#) at the end of Remark 1. $\square$

5 More on identities for trigonometric sums

Actually the literature on identities for finite trigonometric sums is huge, from ancient references like [45, 22, 50] to papers published this year, but here we have only scratched the surface of this vast literature: transitively searching through the references that we give, as well as tracking other papers (that we have skipped because they often deal with weighted sums) with, e.g., the keyword "cotangent sums", yield a lot of many other interesting items. We will only cite [23] (and the references therein).

Such trigonometric sums occur in a large variety of domains, and are addressed with a large variety of methods: compiling all the pertinent papers in the domain would lead at least to a book (look for example at the long list of references given in [5]). Various methods have been used, some of them in papers cited above: *interpolation formulas* [46, 47, 2, 33] (to which we can add, e.g., [10, 1]...), *Ramanujan's theta function* [7, 32, 51], *calculus of residues* (see, among several other papers, the papers by Cvijović or Cvijović et al. cited in [14]; also see [30]), *expansions in partial fractions* (see, e.g., [13, 53, 12]), *discrete Fourier analysis* (see [3]), etc.

The applications go from *number theory* (see, e.g., [10], but also papers dealing with Dedekind sums, generalizations and associated reciprocity laws (see, e.g., [57, 26, 6]...) to *physics* (see, in particular, [20, 21, 15, 16] and the references therein; also see [36]), from *enumerative combinatorics* (e.g., number of closed walks on a path as in [18]) to *binomial identities* (see, e.g. [45, 40, 41, 42, 36]), and to *topology* (see, e.g., [37]), etc.

To end this section, we would like to cite two more results. The first one is a relation between Dirichlet series with trigonometric coefficients and finite trigonometric sums given, among other results, in a paper of J. Franke [25].

Theorem ([25], particular case) The following relation holds

$$\sum_{\substack{n>0 \\ N\nmid n}} \frac{\cot^2(\frac{n\pi}{N})}{n^2} = \frac{(N-1)(N-2)(N^2+3N+2)\pi^2}{90N^2}.$$

The second one, that we find particularly nice, is a result of V. Shevelev [48] and V. Shevelev and P. J. C. Moses [49], linking tangent power sums and sums of digits of integers in integer bases.

Theorem ([48, 49]) *Let n be an odd integer ≥ 3. Let $s_{n-1}(r)$ be the sum of digits of the integer r in base $n-1$. Define S_n by*

$$S_n(x) = \sum_{\substack{0\leq r<x \\ r\equiv 0 \bmod n}} (-1)^{s_{n-1}(r)}.$$

Then if p is a positive integer, one has

$$S_n((n-1)^{2p}) = \frac{2}{n} \sum_{k=1}^{\frac{n-1}{2}} \tan^{2p}\left(\frac{\pi k}{n}\right) \sim \frac{2}{n}(n-1)^{2p\lambda_n} \quad \textit{as } p \textit{ tends to infinity,}$$

where $\lambda_n = \dfrac{\log\cot(\frac{\pi}{2n})}{\log(n-1)}$.

Remark 6 This last result shows why $\sum_{k=1}^{\frac{n-1}{2}} \tan^{2p}\left(\frac{\pi k}{n}\right)$ is an integer multiple of n. Furthermore, for $n=3$, the reader can see that $\lambda_3 = \frac{\log 3}{\log 4}$ and guess that there is a nice relation to the Moser-Newman statement which observes then quantifies that, among the integers multiple of 3, there are "more" evil than odious numbers (recall that a number is said to be *evil* if the sum of its binary digits is even and *odious* otherwise): everything is explained in [48, 49] and the references therein.

6 Proving and reproving

As indicated in [5] *This history* [the history of evaluations and reciprocity theorems for trigonometric sums] *is, not surprisingly, sporadic, and consequently authors often publish results without being aware that their theorems had previously been published elsewhere.* Given that this was written 20 years ago, it should not be a surprise that more recent papers give examples strengthening this remark. We give two more examples: first, compare, say, the abstract of [29] with Section 3 of [33] (or even with the abstract of [28]); second, we will briefly look at some of the results obtained in [51, Theorem 2] via identities for the Ramanujan theta function, and show that they are either in the literature or they can be readily obtained from results already in the literature. First we state an easy result.

Theorem 1 *We have, for n odd, the following identities.*

$$\sum_{k=1}^{\frac{n-1}{2}} (-1)^{k+1} \frac{\sin \frac{2k\pi}{n}}{\sin \frac{k\pi}{n}} = 1$$

and

$$\sum_{k=1}^{\frac{n-1}{2}} (-1)^{k+1} \frac{\sin \frac{k\pi}{n}}{\sin \frac{2k\pi}{n}} = (-1)^{\frac{n+1}{2}} \left(\frac{n-1}{4}\right) + \frac{\chi_o\left(\frac{n-1}{2}\right)}{2}$$

where $\chi_o(m) = 1$ if m is odd and $\chi_o(m) = 0$ if m is even.

Proof. The first equality is equivalent to: for odd n one has

$$2\sum_{k=1}^{\frac{n-1}{2}} (-1)^{k+1} \cos \frac{k\pi}{n} = 1,$$

which is given Remark 1 above.

The second equality is an easy consequence of the identity (for odd n)

$$\sum_{k=1}^{\frac{n-1}{2}} \frac{(-1)^{k+1}}{\cos \frac{k\pi}{n}} = (-1)^{\frac{n+1}{2}} \left(\frac{n-1}{2}\right) + \chi_o\left(\frac{n-1}{2}\right)$$

which can be found, e.g., in [33] (Identity (3.43)).

Remark 7 The first identity in Theorem 1 above gives the identities (1.1), (1.3), and (1.8) of [51, Theorem 2] (take $n = 7, 13, 17$ and use, if needed, that $\sin(\pi - x) = \sin x$). Note that, as indicated in [51], their identity (1.1) is already in [38]. The second equality in Theorem 1 above gives the identity (1.2) of [51, Theorem 2] (take $n = 7$), which, as indicated in [51], is already in [38].

Other identities of [51, Theorem 2] can be proved in an elementary way: for example identities (1.4) and (1.9), which have the form $\sum \pm(\sin a \ \sin b)/(\sin c \ \sin d)$. We only show how to prove easily (1.4). Let L be the lefthand term of Identity (1.4), i.e.,

$$L := \frac{\sin \frac{4\pi}{13}}{\sin \frac{2\pi}{13}} \frac{\sin \frac{6\pi}{13}}{\sin \frac{3\pi}{13}} - \frac{\sin \frac{2\pi}{13}}{\sin \frac{\pi}{13}} \frac{\sin \frac{3\pi}{13}}{\sin \frac{5\pi}{13}} - \frac{\sin \frac{5\pi}{13}}{\sin \frac{4\pi}{13}} \frac{\sin \frac{\pi}{13}}{\sin \frac{6\pi}{13}}$$

which can also be written, using the relations $\sin x = \sin(\pi - x)$, $\sin(2x) = 2 \sin x \cos x$, $2 \cos x \cos y =$

$\cos(x+y)+\cos(x-y)$ and $\cos(\pi-x) = -\cos x$,

$$
\begin{aligned}
L &:= \frac{\sin\frac{4\pi}{13}}{\sin\frac{2\pi}{13}}\frac{\sin\frac{6\pi}{13}}{\sin\frac{3\pi}{13}} - \frac{\sin\frac{2\pi}{13}}{\sin\frac{\pi}{13}}\frac{\sin\frac{10\pi}{13}}{\sin\frac{5\pi}{13}} - \frac{\sin\frac{8\pi}{13}}{\sin\frac{4\pi}{13}}\frac{\sin\frac{12\pi}{13}}{\sin\frac{6\pi}{13}} \\
&= 4\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} - 4\cos\frac{\pi}{13}\cos\frac{5\pi}{13} - 4\cos\frac{4\pi}{13}\cos\frac{6\pi}{13} \\
&= 2\cos\frac{5\pi}{13} + 2\cos\frac{\pi}{13} - 2\cos\frac{6\pi}{13} - 2\cos\frac{4\pi}{13} - 2\cos\frac{10\pi}{13} - 2\cos\frac{2\pi}{13} \\
&= 2\sum_{k=1}^{6}(-1)^{k+1}\cos\frac{k\pi}{13} = 1 \quad \text{(see again Remark 1 above)}
\end{aligned}
$$

Actually one can note that, going backwards and grouping differently the terms in the last sum just above (possibly using $\cos(\pi-x) = -\cos x$ and $\sin(\pi-x) = \sin x$), yields other identities, for example:

$$
\begin{aligned}
1 &= 2\sum_{k=1}^{6}(-1)^{k+1}\cos\frac{k\pi}{13} \\
&= 2\left(\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}\right) - 2\left(\cos\frac{6\pi}{13}+\cos\frac{2\pi}{13}\right) + 2\left(\cos\frac{5\pi}{13}-\cos\frac{4\pi}{13}\right) \\
&= 2\left(\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}\right) - 2\left(\cos\frac{6\pi}{13}+\cos\frac{2\pi}{13}\right) - 2\left(\cos\frac{8\pi}{13}+\cos\frac{4\pi}{13}\right) \\
&= 4\left(\cos\frac{2\pi}{13}\cos\frac{\pi}{13}\right) - 4\left(\cos\frac{4\pi}{13}\cos\frac{2\pi}{13}\right) - 4\left(\cos\frac{6\pi}{13}\cos\frac{2\pi}{13}\right) \\
&= \frac{\sin\frac{4\pi}{13}}{\sin\frac{2\pi}{13}}\frac{\sin\frac{2\pi}{13}}{\sin\frac{\pi}{13}} - \frac{\sin\frac{8\pi}{13}}{\sin\frac{4\pi}{13}}\frac{\sin\frac{4\pi}{13}}{\sin\frac{2\pi}{13}} - \frac{\sin\frac{12\pi}{13}}{\sin\frac{6\pi}{13}}\frac{\sin\frac{4\pi}{13}}{\sin\frac{2\pi}{13}}
\end{aligned}
$$

which can be "disguised" by replacing $\sin x$ with $\sin(\pi-x)$, or here by simplifying and using $\sin(\pi-x) = \sin x$ again, thus obtaining

$$
\frac{\sin\frac{4\pi}{13}}{\sin\frac{\pi}{13}} - \frac{\sin\frac{8\pi}{13}}{\sin\frac{2\pi}{13}} - \frac{\sin\frac{\pi}{13}}{\sin\frac{6\pi}{13}}\frac{\sin\frac{4\pi}{13}}{\sin\frac{2\pi}{13}} = 1, \text{ etc.}
$$

7 Automated proofs of the above and similar identities

Given the type of identities described above, and the re-discoveries of some of them, it seems desirable to have automated proofs: two examples are [17, 19] where the authors allude to the use of `Mathematica`. But it also seems desirable to have not only automated proofs but also automated "discoveries" (and proofs).

Two hints seem to be in favor of possible automated discoveries-proofs (or "shaloshable discovery-proofs", see [59]; also see [44]) of the above and similar trigonometric identities. The (very vague) first hint is the occurrence of a hypergeometric series in [32, Theorem 2.1]. The second hint is the following identity that can be found, e.g., in [8, p. 184]

$$\sin(n\theta) = 2^{n-1} \prod_{r=0}^{n-1} \sin\left(\theta + \frac{r\pi}{n}\right) \cdot \tag{7}$$

This identity can be used to prove a bunch of identities. For example, taking its logarithmic derivative, then differentiating, yields

$$\frac{n^2}{\sin^2(n\theta)} - \frac{1}{\sin^2(\theta)} = \sum_{r=1}^{n-1} \frac{1}{\sin^2\left(\theta + \frac{r\pi}{n}\right)}.$$

Now, letting θ tend to 0, one obtains the identity of Remark 2. But one way to obtain Identity (7) is to use Cebyshev polynomials. Thus one could imagine that manipulating Cebyshev polynomials and the like with a computer algebra system, conveniently taught, would permit to discover/prove trigonometric identities "automatically".

7.1 Automated proofs of specific finite trigonometric identities

Given a multivariable polynomial P, any identity of the form

$$P\left(\sin\left(\frac{\pi}{n}\right), \sin\left(\frac{2\pi}{n}\right), \ldots, \sin\left(\frac{(n-1)\pi}{n}\right)\right) = 0 \tag{1}$$

for a *specific*, positive integer n, is routinely provable by `Maple`, and (probably) any other computer algebra system. In `Maple` the command is `simplify`.

For example the original 'Morrie's law' [55], that Richard Feynman never forgot (also see [39]),

$$\cos(20^\circ)\ \cos(40^\circ)\ \cos(80^\circ)\ =\ \frac{1}{8}$$

can be automatically done (exactly!) by `Maple`. Just type

```
simplify(cos(Pi/9) * cos(2 * Pi/9) * cos(4 * Pi/9));
```

and you will get right away `1/8`.

Nowadays you do not have to be a Gauss to (rigorously!) prove Gauss' identity at the beginning of this article, just enter, in `Maple`

```
simplify((1 - sqrt(17) + sqrt(34 - 2 * sqrt(17)) + 2 * sqrt(17 + 3*\
      sqrt(17) + sqrt(34 - 2 * sqrt(17)) + 2 * sqrt(34 + 2 * sqrt(17))))/16 - cos(Pi/17));
```

and in one nano-second you would get `0`.

To take another example, to prove the second identity on page 1, type

```
sin(3/7 * Pi) * *2/sin(2/7 * Pi) - sin(2/7 * Pi) * *2/sin(1/7 * Pi) + sin(1/7 * Pi) * *2/sin(3/7 * Pi) :

simplify(%);
```

and you would immediately get `0`.

The way `Maple` does it is to use Euler's formula

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \cdot$$

Then in (1) everything is a polynomial in $z := e^{i\pi/n}$, getting, upon expansion, a polynomial in z and making sure that it is divisible by z^n+1. So numeric computations turn into routine, symbolic, 'high-school algebra' calculations that computer algebra systems excel at.

Nine such identities are proved, by 'advanced' methods in [51]. See the output file

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums14.txt`

for automatic, elementary (*high-school algebra*) proofs of all of them.

7.2 Automated proofs of infinite families of finite trigonometric sums

More interesting are "infinite families" of finite trigonometric sums. We will describe how to automatically derive explicit polynomial expressions to the following eight families.

Below n and k are arbitrary positive integers.

Type Top

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{j\pi}{2n+1}\right).$$

Type Ton

$$\sum_{j=1}^{n} \csc^{2k}\left(\frac{j\pi}{2n+1}\right).$$

Type Tep

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{(2j-1)\pi}{4n}\right).$$

Type Ten

$$\sum_{j=1}^{n} \csc^{2k}\left(\frac{(2j-1)\pi}{4n}\right).$$

Type Uop

$$\sum_{j=1}^{n-1} \sin^{2k}\left(\frac{j\pi}{2n}\right).$$

Type Uon

$$\sum_{j=1}^{n-1} \csc^{2k}\left(\frac{j\pi}{2n}\right).$$

Type Uep

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{(2j-1)\pi}{4n+2}\right).$$

Type Uen

$$\sum_{j=1}^{n} \csc^{2k}\left(\frac{(2j-1)\pi}{4n+2}\right).$$

It turns out that to get closed-form expressions for general n and k for the four types **Top, Tep, Uop, Uep**, (i.e., for sums of **positive** even powers of the sines), one does not need computers. These are easy human exercises. Let us just illustrate it with type **Top**.

Define $z := e^{\frac{i\pi}{2n+1}}$. Note that $z^{2n+1} = -1$. Then $\sin(\frac{j\pi}{2n+1}) = (z^j - z^{-j})/(2i)$. We have:

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{j\pi}{2n+1}\right) = \sum_{j=1}^{n} \left((z^j - z^{-j})/(2i)\right)^{2k} = \frac{(-1)^k}{4^k} \sum_{j=1}^{n} (z^j - z^{-j})^{2k}.$$

By the binomial theorem, this equals

$$\frac{(-1)^k}{4^k} \sum_{j=1}^{n} \sum_{r=0}^{2k} (-1)^r \frac{(2k)!}{r!(2k-r)!} z^{-jr+j(2k-r)} = \frac{(-1)^k}{4^k} \sum_{j=1}^{n} \sum_{r=0}^{2k} (-1)^r \frac{(2k)!}{r!(2k-r)!} z^{2j(k-r)}$$

$$= \frac{(-1)^k}{4^k} \cdot n \cdot (-1)^k \frac{(2k)!}{k!^2} + \frac{(-1)^k}{4^k} \sum_{j=1}^{n} \sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!} (z^{2j(k-r)} + z^{-2j(k-r)})$$

$$= \frac{n}{4^k} \frac{(2k)!}{k!^2} + \frac{(-1)^k}{4^k} \sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!} \sum_{j=1}^{n} (z^{2j(k-r)} + z^{-2j(k-r)}).$$

Note that, by summing the geometric series and using $z^{2(2n+1)} = 1$, we have:

$$\sum_{j=1}^{n} (z^{2j(k-r)} + z^{-2j(k-r)}) = \sum_{j=-n}^{n} (z^{2(k-r)})^j - 1 = 0 - 1 = -1.$$

Going back to the above, we have that our desired sum equals

$$= \frac{n}{4^k} \frac{(2k)!}{k!^2} + \frac{(-1)^{k+1}}{4^k} \sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!}.$$

But (you prove it!)

$$\sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!} = \frac{1}{2} (-1)^{k+1} \frac{(2k)!}{k!^2}.$$

We have just proved, *sans ordinateurs*, the general identity:

Theorem Top

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{j\,\pi}{2n+1}\right) = \frac{1}{4^k} \frac{(2k)!}{k!^2} \left(n + \frac{1}{2}\right).$$

Similar things can be done with the other three positive families, and they are left to the reader.

Theorem Tep

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{(2j-1)\pi}{4n}\right) = \frac{n}{2^{2k-1}} \frac{(2k-1)!}{(k-1)!k!}.$$

Theorem Uop

$$\sum_{j=1}^{n-1} \sin^{2k}\left(\frac{j\pi}{2n}\right) \;=\; \frac{n}{4^k}\frac{(2k)!}{k!^2} - \frac{1}{2}.$$

Theorem Uep

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{(2j-1)\pi}{4n+2}\right) \;=\; \frac{n}{4^k}\frac{(2k)!}{k!^2} + \frac{1}{4^k}\frac{(2k-1)!}{k!(k-1)!} - \frac{1}{2}.$$

Note that in the above four theorems we have **closed-form** expressions for *general* n **and** *general* k.

Much more interesting is to derive expressions for the sum of the even powers of the **cosecants** (i.e., reciprocals of the sines).

It turns out that for the case **Uon** Chu and Marini [13] (bottom of Page 126) found an elegant, but very complicated, explicit formula that yields, for each specific k, a polynomial expression, in n, of degree $2k$. Our approach, to be described shortly, while not 'explicit' (it uses recursion), does the same, and seems far more efficient. Chu and Marini also gave an elegant generating function, whose output agrees with ours.

Everything is implemented in the `Maple` package `TrigSums.txt` available, along with numerous input and output files from

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trig.html`.

Remark 8 The results for $k \leq 60$ obtained through this package are given at

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums10.txt`

(also see **Type Uon** in the Appendix) while, using the generating function of [13], the results are given at

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums10ChuMarini.txt`

which takes about the same time: note that, using the explicit formula in [13] is much slower.

We will illustrate our approach with the family of type **Ton**. Recall that the **Chebyshev polynomial of the first kind**, $T_n(x)$ may be defined by

$$T_n(\cos t) = \cos(nt).$$

It is well-known and easy to see [54], that

$$T_n(x) \;=\; \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k 2^{n-2k}(n-k-1)!}{k!(n-2k)!} x^{n-2k}.$$

Hence

$$\frac{T_{2n+1}(x)}{x} \;=\; (2n+1)\sum_{k=0}^{n} \frac{(-1)^k 4^{n-k}(2n-k)!}{k!(2n+1-2k)!}(x^2)^{n-k}.$$

Define the degree n polynomial

$$E_n(x) := \frac{T_{2n+1}(\sqrt{x})}{\sqrt{x}}.$$

Then

$$E_n(x) \;=\; (2n+1)\sum_{k=0}^{n} \frac{(-1)^k 4^{n-k}(2n-k)!}{k!(2n+1-2k)!}x^{n-k}.$$

Note that the n roots of $E_n(x)$ are $\sin^2(\frac{j\,\pi}{2n+1})$, for $j=1,\ldots,n$. Define the *reciprocal polynomial* $\bar{E}_n(x) := x^n E_n(x^{-1})$. Then we have

$$\bar{E}_n(x) \;=\; (-1)^n\,(2n+1)\sum_{k=0}^{n} \frac{(-1)^k 4^k (n+k)!}{(n-k)!(2k+1)!}x^{n-k}.$$

Note that the n roots of $\bar{E}_n(x)$ are $\csc^2(\frac{j\,\pi}{2n+1})$. It follows immediately that, denoting, as usual, the degree k elementary symmetric function of $\alpha_1,\ldots,\alpha_n$ by $e_k(\alpha_1,\ldots,\alpha_n)$, that

Lemma Top

$$e_k(\{\sin^2(j\pi/(2n+1)), j=1\ldots n\}) \;=\; \frac{4^{-k}\,(2n-k)!\,(2n+1)}{k!\,(2n-2k+1)!}.$$

Lemma Ton

$$e_k(\{\csc^2(j\pi/(2n+1)), j=1\ldots n\}) \;=\; \frac{(n+k)!4^k}{(n-k)!\,(2k+1)!}.$$

Now we use Newton's identities [56] (see [58] for a lovely combinatorial proof). Let

$$p_k(\alpha_1,\ldots,\alpha_n) := \sum_{j=1}^{n} \alpha_j^k$$

be the *power-sum functions*, then

$$ke_k(\alpha_1,\ldots,\alpha_n)) \;=\; \sum_{i=1}^{k}(-1)^{i-1}e_{k-i}(\alpha_1,\ldots,\alpha_n)p_i(\alpha_1,\ldots,\alpha_n).$$

This enables us (and our computers) to **recursively** find explicit expressions for the power sums of both $\{\sin^2(j\pi/(2n+1)), j=1\ldots n)\}$ and $\{\csc^2(j\pi/(2n+1)), j=1\ldots n)\}$. Of course, for the former case we do not need it, since we have Theorem **Top** above but it is still nice to know that it agrees up to $k=60$.

For types **Tep** and **Ten**, the underlying polynomial is $T_{2n}(\sqrt{x})$ whose roots are

$$\{\sin^2((2j+1)\pi/(4n)), j=1\ldots n\},$$

and we have, analogously

Lemma Tep

$$e_k(\{\sin^2((2j+1)\,\pi/(4n)), j=1\ldots n\}) \;=\; \frac{2n\,4^{-k}\,(2n-k-1)!}{k!\,(2n-2k)!}.$$

Lemma Ten

$$e_k(\{\csc^2((2j+1)\,\pi/(4n)), j=1\ldots n\}) \;=\; \frac{n\,(n+k-1)!4^k}{(n-k)!\,(2k)!}.$$

For types **Uop** and **Uon**, the underlying polynomial is $U_{2n-1}(\sqrt{x})/\sqrt{x}$ whose roots are

$$\{\sin^2(j\pi/(2n)), j = 1 \ldots n-1\},$$

and we have, analogously

Lemma Uop

$$e_k(\{\sin^2(j\pi/(2n)), j = 1 \ldots n-1\}) \;=\; \frac{4^{-k}\,(2n-k-1)!}{(2n-2k-1)!k!}.$$

Lemma Uon

$$e_k(\{\csc^2(j\pi/(2n)), j = 1 \ldots n-1\}) \;=\; \frac{4^{k}\,(n+k)!\,(n-1)!}{(2k+1)!\,(n-k-1)!n!}.$$

For types **Uep** and **Uen**, the underlying polynomial is $U_{2n}(\sqrt{x})$ whose roots are

$$\{\sin^2((2j-1)\pi/(4n+2)), j = 1 \ldots n\},$$

and we have, analogously

Lemma Uep

$$e_k(\{\sin^2((2j-1)\,\pi/(4n+2)), j = 1 \ldots n\}) \;=\; \frac{4^{-k}\,(2n-k)!}{(2n-2k)!k!}.$$

Lemma Uen

$$e_k(\{\csc^2((2j-1)\pi/(4n+2)), j = 1 \ldots n\}) \;=\; \frac{(n+k)!4^{k}}{(2k)!\,(n-k)!}.$$

From these, combined with Netwon's identities one can find as many sum-of-powers as one wishes.

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I. M. Gessel, Generating functions and generalized Dedekind sums, in The Wilf Festschrift (Philadelphia, PA, 1996), *Electron. J. Combin.* **4** (1997), Research Paper 11, 17 pp.

C. A. Cadavid, P. Hoyos, J. Jorgenson, L. Smajlović, J. D. Vélez, On an approach for evaluating certain trigonometric character sums using the discrete time heat kernel, Preprint (2022), `https://arxiv.org/abs/2201.07878`.

Appendix by Shalosh B. Ekhad

Type Ton

Proposition $\mathbf{Ton_1}$

$$\sum_{j=1}^{n} \csc^2\left(\frac{\pi j}{2n+1}\right) = \frac{2(n+1)n}{3}.$$

Proposition $\mathbf{Ton_2}$

$$\sum_{j=1}^{n} \csc^4\left(\frac{\pi j}{2n+1}\right) = \frac{8(n+1)n\left(n^2+n+3\right)}{45}$$

Proposition $\mathbf{Ton_3}$

$$\sum_{j=1}^{n} \csc^6\left(\frac{\pi j}{2n+1}\right) = \frac{8(n+1)n\left(8n^4+16n^3+35n^2+27n+54\right)}{945}.$$

Proposition $\mathbf{Ton_4}$

$$\sum_{j=1}^{n} \csc^8\left(\frac{\pi j}{2n+1}\right) = \frac{128(n+1)n\left(n^2+n+3\right)\left(3n^4+6n^3+7n^2+4n+15\right)}{14175}.$$

Proposition $\mathbf{Ton_5}$

$$\sum_{j=1}^{n} \csc^{10}\left(\frac{\pi j}{2n+1}\right) = \frac{64(n+1)n\left(16n^8+64n^7+182n^6+322n^5+493n^4+524n^3+579n^2+360n+540\right)}{93555}.$$

For Propositions $\mathbf{Ton_k}$ for $6 \le k \le 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums1.txt`.

Type Ten

Proposition $\mathbf{Ten_1}$

$$\sum_{j=1}^{n} \csc^2\left(\frac{\pi(2j-1)}{4n}\right) = 2n^2.$$

Proposition $\mathbf{Ten_2}$

$$\sum_{j=1}^{n} \csc^4\left(\frac{\pi(2j-1)}{4n}\right) = \frac{4n^2\left(2n^2+1\right)}{3}.$$

Proposition $\mathbf{Ten_3}$

$$\sum_{j=1}^{n} \csc^6\left(\frac{\pi(2j-1)}{4n}\right) = \frac{8n^2\left(8n^4+5n^2+2\right)}{15}.$$

Proposition $\mathbf{Ten_4}$

$$\sum_{j=1}^{n} \csc^8\left(\frac{\pi(2j-1)}{4n}\right) = \frac{16n^2\left(136n^6+112n^4+49n^2+18\right)}{315}.$$

Proposition $\mathbf{Ten_5}$

$$\sum_{j=1}^{n} \csc^{10}\left(\frac{\pi(2j-1)}{4n}\right) = \frac{32n^2\left(992n^8+1020n^6+546n^4+205n^2+72\right)}{2835}.$$

For Propositions $\mathbf{Ten_k}$ for $6 \le k \le 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums5.txt`.

Type Uon

Proposition $\mathbf{Uon_1}$

$$\sum_{j=1}^{n-1} \csc^{2}\left(\frac{j\pi}{2n}\right) = \frac{2\,(n+1)\,(n-1)}{3}.$$

Proposition $\mathbf{Uon_2}$

$$\sum_{j=1}^{n-1} \csc^{4}\left(\frac{j\pi}{2n}\right) = \frac{4\,(n+1)\,(n-1)\left(2n^2+7\right)}{45}.$$

Proposition $\mathbf{Uon_3}$

$$\sum_{j=1}^{n-1} \csc^{6}\left(\frac{j\pi}{2n}\right) = \frac{8\,(n+1)\,(n-1)\left(8n^4+29n^2+71\right)}{945}.$$

Proposition $\mathbf{Uon_4}$

$$\sum_{j=1}^{n-1} \csc^{8}\left(\frac{j\pi}{2n}\right) = \frac{16\,(n+1)\,(n-1)\left(24n^6+104n^4+251n^2+521\right)}{14175}.$$

Proposition $\mathbf{Uon_5}$

$$\sum_{j=1}^{n-1} \csc^{10}\left(\frac{j\pi}{2n}\right) = \frac{32\,(n+1)\,(n-1)\left(32n^8+164n^6+450n^4+901n^2+1693\right)}{93555}.$$

For Propositions $\mathbf{Uon_k}$ for $6 \le k \le 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums9.txt`.

Type Uen

Proposition $\mathbf{Uen_1}$

$$\sum_{j=1}^{n} \csc^{2}\left(\frac{(2j-1)\pi}{4n+2}\right) = 2\,(n+1)\,n.$$

Proposition $\mathbf{Uen_2}$

$$\sum_{j=1}^{n} \csc^{4}\left(\frac{(2j-1)\pi}{4n+2}\right) = \frac{8\,(n+1)\,n\left(n^2+n+1\right)}{3}.$$

Proposition $\mathbf{Uen_3}$

$$\sum_{j=1}^{n} \csc^{6}\left(\frac{(2j-1)\pi}{4n+2}\right) = \frac{8\,(n+1)\,n\left(8n^4+16n^3+19n^2+11n+6\right)}{15}.$$

Proposition $\mathbf{Uen_4}$

$$\sum_{j=1}^{n} \csc^{8}\left(\frac{(2j-1)\pi}{4n+2}\right) = \frac{128\,(n+1)\,n\,\left(n^2+n+1\right)\left(17n^4+34n^3+31n^2+14n+9\right)}{315}.$$

Proposition $\mathbf{Uen_5}$

$$\sum_{j=1}^{n} \csc^{10}\left(\frac{(2j-1)\pi}{4n+2}\right)$$

$$= \frac{64\,(n+1)\,n\,\left(496n^8+1984n^7+4106n^6+5374n^5+4979n^4+3316n^3+1669n^2+576n+180\right)}{2835}.$$

For Propositions $\mathbf{Uen_k}$ for $6 \le k \le 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums12.txt`.

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