**Automated part of the Allouche-Zeilberger paper**

**Note: This is a Plain TeX draft that will be incorporated in the main paper written in LaTex. Only for the use of J. -P. Allouche**

**Automated proofs of Specific Finite Trigonometric Identities**

Given a multivariable polynomial $P$, any identity of the form,

$$P(\sin(\frac{\pi}{n}), \sin(\frac{2\pi}{n}), \dots, sin(\frac{(n-1)\pi}{n})) = 0 \quad , \tag{1}$$

for a *specific*, positive integer $n$, is routinely provable by Maple, and (probably) any other computer algebra system. In Maple the command is `simplify`.

For example the original 'Morrie's law'[4], that Richard Feynman never forgot (also see [1])

$$cos(20^o)\, cos(40^o)\, cos(80^o) \;=\; \frac{1}{8} \quad ,$$

can be automatically done (exactly!) by Maple. Just type

```
simplify(cos(Pi/9)*cos(2*Pi/9)*cos(4*Pi/9));
```

and you will get right away `1/8`.

Nowadays you don't have to be a Gauss to (rigorously!) prove Gauss' identity at the beginning of this article, just enter, in Maple

```
simplify((-1+sqrt(17)+sqrt(34-2*sqrt(17))+2*sqrt(17+3*sqrt(17)-sqrt(170+38*sqrt(17))))/16-
cos(2*Pi/17));
```

and in one nano-second you would get `0`.

To take another example, to prove the second identity on page 1, type

```
sin(3/7*Pi)**2/sin(2/7*Pi) - sin(2/7*Pi)**2/sin(1/7*Pi) + sin(1/7*Pi)**2/sin(3/7*Pi):

simplify(%);
```

,

and you would immediately get `0`.

The way Maple does it is to use Euler's formula

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad .$$

Then in (1) everything is a polynomial in $z := e^{i\pi/n}$, getting, upon expansion, a polynomial in $z$ and making sure that it is divisible by $z^n+1$. So numeric computations turn into routine, symbolic, 'high-school algebra' calculations that computer algebra systems excel at.

Nine such identities are proved, by 'advanced' methods in [2]. See the output file

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums14.txt` ,

for automatic, elementary (*high-school algebra*) proofs of all of them.

**Automated Proofs of Infinite Families of Finite Trigonometric Sums**

More interesting are "infinite families" of finite trigonometric sums. We will describe how to automatically derive explicit polynomial expressions to the following eight families.

Below $n$ and $k$ are arbitrary positive integers.

**Type Top**

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{j\,\pi}{2n+1}\right) \quad .$$

**Type Ton**

$$\sum_{j=1}^{n} \csc^{2k}\left(\frac{j\,\pi}{2n+1}\right) \quad .$$

**Type Tep**

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{(2j-1)\,\pi}{4n}\right) \quad .$$

**Type Ten**

$$\sum_{j=1}^{n} \csc^{2k}\left(\frac{(2j-1)\,\pi}{4n}\right) \quad .$$

**Type Uop**

$$\sum_{j=1}^{n-1} \sin^{2k}\left(\frac{j\,\pi}{2n}\right) \quad .$$

**Type Uon**

$$\sum_{j=1}^{n-1} \csc^{2k}\left(\frac{j\,\pi}{2n}\right) \quad .$$

**Type Uep**

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{(2j-1)\,\pi}{4n+2}\right) \quad .$$

**Type Uen**

$$\sum_{j=1}^{n} \csc^{2k}\left(\frac{(2j-1)\,\pi}{4n+2}\right) \quad .$$

It turns out that to get closed-form expressions for general $n$ and $k$ for the four types **Top,Tep,Uop, Uep**, (i.e. for sums of **positive** even powers of the sines), one does not need computers. These are easy human exercises. Let's just illustrate it with type **Top**.

Define $z := e^{\frac{i\pi}{2n+1}}$. Note that $z^{2n+1} = -1$. Then $\sin(\frac{j\,\pi}{2n+1}) = (z^j - z^{-j})/(2i)$. We have:

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{j\,\pi}{2n+1}\right) = \sum_{j=1}^{n} \left((z^j - z^{-j})/(2i)\right)^{2k} = \frac{(-1)^k}{4^k} \sum_{j=1}^{n} (z^j - z^{-j})^{2k} \quad .$$

By the binomial theorem, this equals

$$\frac{(-1)^k}{4^k} \sum_{j=1}^{n} \sum_{r=0}^{2k} (-1)^r \frac{(2k)!}{r!(2k-r)!} z^{-jr+j(2k-r)} = \frac{(-1)^k}{4^k} \sum_{j=1}^{n} \sum_{r=0}^{2k} (-1)^r \frac{(2k)!}{r!(2k-r)!} z^{2j(k-r)}$$

$$= \frac{(-1)^k}{4^k} \cdot n \cdot (-1)^k \frac{(2k)!}{k!^2} + \frac{(-1)^k}{4^k} \sum_{j=1}^{n} \sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!} (z^{2j(k-r)} + z^{-2j(k-r)})$$

$$= \frac{n}{4^k} \frac{(2k)!}{k!^2} + \frac{(-1)^k}{4^k} \sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!} \sum_{j=1}^{n} (z^{2j(k-r)} + z^{-2j(k-r)}) \quad .$$

Note that, by summing the geometric series and using $z^{2(2n+1)} = 1$, we have:

$$\sum_{j=1}^{n} (z^{2j(k-r)} + z^{-2j(k-r)}) = \sum_{j=-n}^{n} (z^{2(k-r)})^j - 1 = 0 - 1 = -1 \quad .$$

Going back to the above, we have that our desired sum equals

$$= \frac{n}{4^k} \frac{(2k)!}{k!^2} + \frac{(-1)^{k+1}}{4^k} \sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!} \quad .$$

But (you prove it!)

$$\sum_{r=0}^{k-1} (-1)^r \frac{(2k)!}{r!(2k-r)!} = \frac{1}{2} (-1)^{k+1} \frac{(2k)!}{k!^2} \quad .$$

We have just proved, *sans ordinateurs*, the general identity.

**Theorem Top**:

$$\sum_{j=1}^{n} \sin^{2k}\left(\frac{j\,\pi}{2n+1}\right) = \frac{1}{4^k} \frac{(2k)!}{k!^2} \left(n + \frac{1}{2}\right) \quad .$$

Similar things can be done with the other three positive families, and they are left to the reader.

**Theorem Tep**:

$$\sum_{j=1}^{n} \sin^{2k}(\frac{(2j-1)\,\pi}{4n}) \;=\; \frac{n}{2^{2k-1}}\,\frac{(2k-1)!}{(k-1)!k!} \quad .$$

**Theorem Uop**:

$$\sum_{j=1}^{n-1} \sin^{2k}(\frac{j\,\pi}{2n}) \;=\; \frac{n}{4^k}\,\frac{(2k)!}{k!^2} - \frac{1}{2} \quad .$$

**Theorem Uep**:

$$\sum_{j=1}^{n} \sin^{2k}(\frac{(2j-1)\,\pi}{4n+2}) \;=\; \frac{n}{4^k}\,\frac{(2k)!}{k!^2} + \frac{1}{4^k}\,\frac{(2k-1)!}{k!(k-1)!} - \frac{1}{2} \quad .$$

Note that in the above four theorems we have **closed-form** expressions for *general* $n$ **and** *general* $k$.

Much more interesting is to derive expressions for the sum of the even powers of the **cosecants** (i.e. reciprocals of the sines). In this case there does not seem to be a general formula for $n$ **and** $k$, but for each specific $k$, one can get an explicit expression, as a polynomial in $n$ of degree $2k$, for any desired $k$. We went as far as $k \leq 60$, but readers are welcome to use our accompanying Maple package `TrigSums.txt`, available from the front of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trig.html` ,

to get as far as one wishes.

We will illustrate it with the family of type **Ton**.

Recall that the **Chebyshev polynomial of the first kind**, $T_n(x)$ may be defined by

$$T_n(\cos t) = cos(nt) \quad .$$

It is well-known and easy to see [3], that

$$T_n(x) \;=\; \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k 2^{n-2k}(n-k-1)!}{k!(n-2k)!} x^{n-2k} \quad .$$

Hence

$$\frac{T_{2n+1}(x)}{x} \;=\; (2n+1)\sum_{k=0}^{n} \frac{(-1)^k 4^{n-k}(2n-k)!}{k!(2n+1-2k)!}(x^2)^{n-k} \quad .$$

Define the degree $n$ polynomial

$$E_n(x) := \frac{T_{2n+1}(\sqrt{x})}{\sqrt{x}} \quad .$$

Then

$$E_n(x) \,=\, (2n+1)\sum_{k=0}^{n} \frac{(-1)^k 4^{n-k}(2n-k)!}{k!(2n+1-2k)!} x^{n-k} \quad .$$

Note that the $n$ roots of $E_n(x)$ are $\sin^2(\frac{j\,\pi}{2n+1})$, for $j = 1, \dots, n$.

Define the *reciprocal polynomial* $\bar{E}_n(x) := x^n E_n(x^{-1})$. Then we have

$$\bar{E}_n(x) \,=\, (-1)^n\,(2n+1)\sum_{k=0}^{n} \frac{(-1)^k 4^k (n+k)!}{(n-k)!(2k+1)!} x^{n-k} \quad .$$

Note that the $n$ roots of $\bar{E}_n(x)$ are $\csc^2(\frac{j\,\pi}{2n+1})$.

It follows immediately that, denoting, as usual, the degree $k$ elementary symmetric function of of $\alpha_1, \dots, \alpha_n$ by $e_k(\alpha_1, \dots, \alpha_n)$, that

**Lemma Top**:

$$e_k(\{sin^2(j\,\pi/(2n+1), j = 1 \dots n)\}) \,=\, \frac{4^{-k}\,(2n-k)!\,(2n+1)}{k!\,(2n-2k+1)!}$$

**Lemma Ton**:

$$e_k(\{\csc^2(j\,\pi/(2n+1)), j = 1 \dots n)\}) \,=\, \frac{(n+k)!4^k}{(n-k)!\,(2k+1)!} \quad .$$

Now we use Newton's identities [5] (see [6] for a lovely combinatorial proof). Let

$$p_k(\alpha_1, \dots, \alpha_n) := \sum_{j=1}^{n} \alpha_j^k$$

be the *power-sum functions*, then

$$ke_k(\alpha_1, \dots, \alpha_n)) \,=\, \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}(\alpha_1, \dots, \alpha_n) p_i(\alpha_1, \dots, \alpha_n) \quad .$$

This enables us (and our computers) to **recursively** find explicit expressions for the power sums of both $\{sin^2(j\,\pi/(2n+1)), j = 1 \dots n)\}$ and $\{csc^2(j\,\pi/(2n+1)), j = 1 \dots n)\}$. Of course, for the former case we don't need it, since we have Theorem **Top** above but it is still nice to know that it agrees up to $k = 60$.

For types **Tep** and **Ten**, the underlying polynomial is $T_{2n}(\sqrt{x})$ whose roots are

$$\{\sin^2((2j+1)\pi/(4n)), j = 1 \ldots n)\} \quad ,$$

and we have, analogously

**Lemma Tep**:

$$e_k(\{\sin^2((2j+1)\,\pi/(4n)), j = 1 \ldots n\}) \;=\; \frac{2n\,4^{-k}\,(2n-k-1)!}{k!\,(2n-2k)!} \quad .$$

**Lemma Ten**:

$$e_k(\{\csc^2((2j+1)\,\pi/(4n)), j = 1 \ldots n\}) \;=\; \frac{n\,(n+k-1)!4^k}{(n-k)!\,(2k)!} \quad .$$

For types **Uop** and **Uon**, the underlying polynomial is $U_{2n-1}(\sqrt{x})/\sqrt{x}$ whose roots are

$$\{\sin^2(j\pi/(2n)), j = 1 \ldots n-1)\}$$

and we have, analogously

**Lemma Uop**:

$$e_k(\{\sin^2(j\,\pi/(2n)), j = 1 \ldots n-1\}) \;=\; \frac{4^{-k}\,(2n-k-1)!}{(2n-2k-1)!k!} \quad .$$

**Lemma Uon**:

$$e_k(\{\csc^2(j\,\pi/(2n)), j = 1 \ldots n-1\}) \;=\; \frac{4^k\,(n+k)!\,(n-1)!}{(2k+1)!\,(n-k-1)!n!} \quad .$$

For types **Uep** and **Uen**, the underlying polynomial is $U_{2n}(\sqrt{x})$ whose roots are

$$\{\sin^2((2j-1)\pi/(4n+2), j = 1 \ldots n)\} \quad ,$$

and we have, analogously

**Lemma Uep**:

$$e_k(\{\sin^2((2j-1)\,\pi/(4n+2)), j=1\dots n\}) \;=\; \frac{4^{-k}\,(2n-k)!}{(2n-2k)!k!} \quad ,$$

**Lemma Uen**:

$$e_k(\{\csc^2((2j-1)\,\pi/(4n+2)), j=1\dots n\}) \;=\; \frac{(n+k)!4^k}{(2k)!\,(n-k)!} \quad .$$

From these, combined with Netwon's identities one can find as many sum-of-powers as one wishes.

**Appendix by Shalosh B. Ekhad**

**Type Ton**

**Proposition Ton[1]**

$$\sum_{j=1}^{n} \csc^2(\frac{\pi j}{2n+1}) \;=\; \frac{2\,(n+1)\,n}{3} \quad .$$

**Proposition Ton[2]**

$$\sum_{j=1}^{n} \csc^4(\frac{\pi j}{2n+1}) \;=\; \frac{8\,(n+1)\,n\,\left(n^2+n+3\right)}{45}$$

**Proposition Ton[3]**

$$\sum_{j=1}^{n} \csc^6(\frac{\pi j}{2n+1}) \;=\; \frac{8\,(n+1)\,n\,\left(8n^4+16n^3+35n^2+27n+54\right)}{945} \quad .$$

**Proposition Ton[4]**

$$\sum_{j=1}^{n} \csc^8(\frac{\pi j}{2n+1}) \;=\; \frac{128\,(n+1)\,n\,\left(n^2+n+3\right)\left(3n^4+6n^3+7n^2+4n+15\right)}{14175} \quad .$$

**Proposition Ton[5]**

$$\sum_{j=1}^{n} \csc^{10}(\frac{\pi j}{2n+1}) \;=\; \frac{64\,(n+1)\,n\,\left(16n^8+64n^7+182n^6+322n^5+493n^4+524n^3+579n^2+360n+540\right)}{93555} \quad .$$

For Propositions **Ton[k]** for $6 \le k \le 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums1.txt` .

**Type Ten**

**Proposition Ten[1]**

$$\sum_{j=1}^{n} \csc^2(\frac{\pi(2j-1)}{4n}) \;=\; 2\,n^2 \quad .$$

**Proposition Ten[2]**

$$\sum_{j=1}^{n} \csc^4(\frac{\pi(2j-1)}{4n}) \;=\; \frac{4n^2\left(2n^2+1\right)}{3} \quad .$$

**Proposition Ten[3]**

$$\sum_{j=1}^{n} \csc^6(\frac{\pi(2j-1)}{4n}) \;=\; \frac{8n^2\left(8n^4+5n^2+2\right)}{15} \quad .$$

**Proposition Ten[4]**

$$\sum_{j=1}^{n} \csc^8(\frac{\pi(2j-1)}{4n}) \;=\; \frac{16n^2\left(136n^6+112n^4+49n^2+18\right)}{315} \quad .$$

**Proposition Ten[5]**

$$\sum_{j=1}^{n} \csc^{10}(\frac{\pi(2j-1)}{4n}) \;=\; \frac{32n^2\left(992n^8+1020n^6+546n^4+205n^2+72\right)}{2835} \quad .$$

For Propositions **Ten[k]** for $6 \leq k \leq 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums5.txt` .

**Type Uon**

**Proposition Uon[1]**

$$\sum_{j=1}^{n-1} \csc^2(\frac{j\,\pi}{2n}) \;=\; \frac{2\,(n+1)\,(n-1)}{3} \quad .$$

**Proposition Uon[2]**

$$\sum_{j=1}^{n-1} \csc^4(\frac{j\,\pi}{2n}) \;=\; \frac{4\,(n+1)\,(n-1)\left(2n^2+7\right)}{45} \quad .$$

**Proposition Uon[3]**

$$\sum_{j=1}^{n-1} \csc^6(\frac{j\,\pi}{2n}) \;=\; \frac{8\,(n+1)\,(n-1)\left(8n^4+29n^2+71\right)}{945} \quad .$$

**Proposition Uon[4]**

$$\sum_{j=1}^{n-1} \csc^8(\frac{j\,\pi}{2n}) \;=\; \frac{16\,(n+1)\,(n-1)\,\left(24n^6+104n^4+251n^2+521\right)}{14175} \quad .$$

**Proposition Uon[5]**

$$\sum_{j=1}^{n-1} \csc^{10}(\frac{j\,\pi}{2n}) \;=\; \frac{32\,(n+1)\,(n-1)\,\left(32n^8+164n^6+450n^4+901n^2+1693\right)}{93555} \quad .$$

For Propositions **Uon[k]** for $6 \le k \le 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums9.txt` .

**Type Uen**

**Proposition Uen[1]**

$$\sum_{j=1}^{n} \csc^2(\frac{(2j-1)\,\pi}{4n+2}) \;=\; 2\,(n+1)\,n \quad .$$

**Proposition Uen[2]**

$$\sum_{j=1}^{n} \csc^4(\frac{(2j-1)\,\pi}{4n+2}) \;=\; \frac{8\,(n+1)\,n\,\left(n^2+n+1\right)}{3} \quad .$$

**Proposition Uen[3]**

$$\sum_{j=1}^{n} \csc^6(\frac{(2j-1)\,\pi}{4n+2}) \;=\; \frac{8\,(n+1)\,n\,\left(8n^4+16n^3+19n^2+11n+6\right)}{15} \quad .$$

**Proposition Uen[4]**

$$\sum_{j=1}^{n} \csc^8(\frac{(2j-1)\,\pi}{4n+2}) \;=\; \frac{128\,(n+1)\,n\,\left(n^2+n+1\right)\left(17n^4+34n^3+31n^2+14n+9\right)}{315} \quad .$$

**Proposition Uen[5]**

$$\sum_{j=1}^{n} \csc^{10}(\frac{(2j-1)\,\pi}{4n+2})$$

$$= \frac{64\,(n+1)\,n\,\left(496n^8+1984n^7+4106n^6+5374n^5+4979n^4+3316n^3+1669n^2+576n+180\right)}{2835} \quad .$$

For Propositions **Uen[k]** for $6 \le k \le 50$, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTrigSums12.txt` .

## References

[1] J. Louck, W. Beyer, and D. Zeilberger, *A generalization of a curiosity that Feynman remembered all his life*, 'Math Bite', Math. Magazine, 69(1)(Feb. 1996), 43-44.
`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/feynman.html`

[2] G. Vinay, H. T. Shwetha and K. N. Harshitha, *Non-trivial trigonometric sums arising from some of Ramanujan Theta function identities*, Palestinian J. Mathematics 11(1)(2022), 130-134.

[3] The Wikipedia Foundation, *Chebyshev polynomials*.

[4] The Wikipedia Foundation, *Morrie's Law* .

[5] The Wikipedia Foundation, *Newton's identities*.

[6] D. Zeilberger, *A combinatorial proof of Newton's identities*, Discrete Mathematics **49** (1984), 319.
`https://sites.math.rutgers.edu/~zeilberg/mamarimY/DZ1984.pdf`