A New Proof that there are 2^n Possible Outcomes on Tossing a Coin n Times

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A short proof is not as satisfying as a long one. A good mathematical proof should do much more than just *prove* the theorem. It should amuse, instruct, and satisfy our deeper love of knowledge, which is much more important than our mere *desire* to know whether the theorem is true.

I find that the standard proof that the number of elements of

$$\mathcal{W}_n := \{w = w_1 \dots w_n \mid w_i = H \quad or \quad T\}$$

equals 2^n is not quite satisfactory. It is based on the very superficial 'structure theorem': $\mathcal{W}_n = \mathcal{W}_{n-1} \times \{H, T\}$ which immediately implies that $|\mathcal{W}_n| = |\mathcal{W}_{n-1}| \cdot 2$, which immediately implies $|\mathcal{W}_n| = 2^n$, by induction.

In this note I will give a new proof of this famous fact, that is based on a much deeper and more interesting structure that the set of sequences of Heads and Tails possesses. As a bonus, we will also get a new proof of Erik Sparre Andersen's famous 'arcsine' result ([F]), that the number of ways of tossing a coin $2n$ times, and having at least as many Heads as Tails at exactly $2k$ of these tosses, is

$$\binom{2n-2k}{n-k}\binom{2k}{k} \quad .$$

The present proof is in the style and spirit of our good master, Marco Schützenberger.

Let $H := +1$ and $T := -1$. Define the two sets

$$\mathcal{C}_n := \{w \in \mathcal{W}_n \mid \sum_{j=1}^{i} w_j \geq 0 \ , \ 1 \leq i < n \, , \quad and \quad \sum_{j=1}^{n} w_j = 0 \ \} \quad ,$$

the famous Dyck words (or Catalan paths), and

$$\mathcal{U}_n := \{w \in \mathcal{W}_n \mid \sum_{j=1}^{i} w_j \geq 0 \ , 1 \leq i \leq n\} \quad .$$

Let

$$\mathcal{W} := \cup_{n=0}^{\infty} \mathcal{W}_n \quad , \quad \mathcal{C} := \cup_{n=0}^{\infty} \mathcal{C}_n \quad , \quad \mathcal{U} := \cup_{n=0}^{\infty} \mathcal{U}_n \quad .$$

For each element w of $\mathcal{W}$, we will define its *weight* to be $weight(w) := z^n$, where n is its length (i.e. $w \in \mathcal{W}_n$). The main theorem of this paper is equivalent to:

$$\alpha(z) := \sum_{w \in \mathcal{W}} weight(w) = \frac{1}{1-2z} \quad ,$$

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as can be seen by extracting the coefficient of z^n from both sides.

It is readily seen (and well known) that any element w of $\mathcal{C}$ is either the empty word (whose weight is 1), or else can be written as $HuTv$, where u and v are shorter members of $\mathcal{C}$. Hence, we have the 'syntax':

$$\mathcal{C} = empty \ \cup \ H\mathcal{C}T\mathcal{C} \quad .$$

Since $weight(HuTv) = z^2 weight(u) weight(v)$, we have that the *total weight*,

$$\psi(z) := \sum_{w \in \mathcal{C}} weight(w) \quad ,$$

satisfies the *quadratic equation*

$$\psi(z) = 1 + z^2 \psi(z)^2 \quad . \qquad (eq1)$$

It is also easy to see that every element w of $\mathcal{U}$ is either the empty word or can be written as Hu, where $u \in \mathcal{U}$, or can be written as $HuTv$, where $u \in \mathcal{C}$ and $v \in \mathcal{U}$. Hence, taking $\mathcal{C}$ as given, the language $\mathcal{U}$ has the syntax:

$$\mathcal{U} = empty \quad \cup \ H\mathcal{U} \ \cup \ H\mathcal{C}T\mathcal{U} \quad .$$

Taking weight, yields that

$$\phi(z) := \sum_{w \in \mathcal{U}} weight(w) \quad ,$$

satisfies the equation

$$\phi(z) = 1 + z\phi(z) + z^2 \psi(z) \phi(z) \quad . \qquad (eq2)$$

Let $\bar{\mathcal{C}}$ and $\bar{\mathcal{U}}$ be the reflections of $\mathcal{C}$ and $\mathcal{U}$ respectively, where the H's and T's are interchanged. It is easy to see that any element $w \in \mathcal{W}$ is either the empty sequence, or can be written as Hu, with $u \in \mathcal{U}$, or $T\bar{u}$, with $\bar{u} \in \bar{\mathcal{U}}$, or $HuTw'$, with $u \in \mathcal{C}$, and $w' \in \mathcal{W}$, or $T\bar{u}Hw'$, with $\bar{u} \in \bar{\mathcal{C}}$, and $w' \in \mathcal{W}$. Hence $\mathcal{W}$ has the following syntax:

$$\mathcal{W} = empty \ \cup \ H\mathcal{U} \ \cup T\bar{\mathcal{U}} \cup \ H\mathcal{C}T\mathcal{W} \cup T\bar{\mathcal{C}}H\mathcal{W} \quad . \qquad (Takhbir)$$

Taking, *weight*, gives the equation:

$$\alpha(z) = 1 + 2z\phi(z) + 2z^2 \psi(z) \alpha(z) \quad . \qquad (eq3)$$

Solving the quadratic equation $(eq1)$, then using $(eq2)$ to get an expression for $\phi(z)$, and then using $(eq3)$ to get an expression for $\alpha(z)$, and simplifying, yields that indeed $\alpha(z) = (1-2z)^{-1}$. $\square$

To get Sparre Andersen's result, define a new weight, *poids*, as follows. First, for $w \in \mathcal{W}_n$, let

$$up(w) := \#\{1 \le i \le n \mid \sum_{j=1}^{i-1} w_j \ge 0 \quad and \sum_{j=1}^{i} w_j \ge 0 \ \} \quad ,$$

the number of tosses for which it was not losing right before and right after it (assuming that you win and lose a dollar according to whether you got H or T). The new weight, *poids*, is defined by:

$$poids(w) := z^n t^{up(w)} \quad ,$$

where n is the length of the word i.e. $w \in \mathcal{W}_n$. Obviously $poids(\mathcal{C}) = \psi(zt)$, $poids(\mathcal{U}) = \phi(zt)$, while $poids(\bar{\mathcal{C}}) = \psi(z)$, $poids(\bar{\mathcal{U}}) = \phi(z)$. Taking *poids* in Eq. $(Takhbir)$, yields that,

$$\beta(z,t) := \sum_{w \in \mathcal{W}} poids(w) \quad ,$$

satisfies the equation:

$$\beta(z,t) = 1 + z\phi(z) + zt\phi(zt) + (z^2\psi(z) + (zt)^2\psi(zt))\beta(z,t) \quad . \qquad (eq4)$$

Solving this linear equation for β, taking the even part, with respect to z:

$$\widetilde{\beta}(z,t) := \frac{\beta(z,t) + \beta(-z,t)}{2} \quad ,$$

and simplifying, yields that

$$\widetilde{\beta}(z,t) = (1 - 4z^2)^{-1/2}(1 - 4z^2t^2)^{-1/2} \quad .$$

Since

$$(1 - 4z^2)^{-1/2} = \sum_{m=0}^{\infty} \binom{2m}{m} z^{2m} \quad ,$$

we see that the number of ways of tossing a coin $2n$ times, and not being at loss in $2k$ of them, i.e. the number of w for which $poids(w) = z^{2n}t^{2k} = z^{2(n-k)}(zt)^{2k}$, which is the coefficient of $z^{2(n-k)}(zt)^{2k}$ in

$$\widetilde{\beta}(z,t) = (1 - 4z^2)^{-1/2} \cdot (1 - 4z^2t^2)^{-1/2} \quad ,$$

is indeed

$$\binom{2n-2k}{n-k}\binom{2k}{k} \quad ,$$

as proved first by Sparre Andersen.

A short Maple program `sparre` that carries all the routine verifications of this paper, is available by anonymous `ftp` to `ftp.math.temple.edu`, in directory `/pub/zeilberg/stam`. The output file `outsparre` can also be found there.

Thanks

I was reminded about this beautiful result of Sparre Andersen two days ago, when I attended a fascinating 20-minute talk, about the arcsine law, by my colleague Omar Hijab (`hijab@math.temple.edu http://www.math.temple.edu/~hijab`). His talk, intended for undergraduates, was part of the festivities of the Temple University Math Day, organized by Hijab. I

strongly recommend that you read, and consider adopting, his excellent forthcoming textbook [H], that contains, among much other fascinating material, a discussion of the arcsine law.

References

[F] W. Feller, *An Introduction to Probability Theory and Its Applications*, volume 1. Wiley, 1950; third edition, 1968.

[H] O. Hijab, *A Course in Real Analysis*, Springer, to appear.