Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms and the Sharpening of Wilf-Zeilberger Theory

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Sharp Upper Bounds for the Almkvist-Zeilberger Algorithm

This section is a discrete-continuous analog of [MZ].

Theorem Let

$$F(n,x) = POL(n,x) \cdot H(n,x) \quad , \qquad (DiscreteContHyperGeometric)$$

where $POL(n,x)$ is a polynomial of (n,x), and

$$H(n,x) = e^{a(x)/b(x)} \cdot \left(\prod_{p=1}^{P} S_p(x)^{\alpha_p} \right) \cdot \left(\frac{s(x)}{t(x)} \right)^n \qquad (PureDiscreteContHyperGeometric)$$

where $a(x), b(x), s(x), t(x)$ and $S_p(x)$ $(1 \leq p \leq P)$ are *polynomials* of x, while the α_p's are commuting *indeterminates*, Let

$$L = deg(b) + deg(s) + deg(t) + \sum_{p=1}^{P} deg(S_p) + max(deg(a), deg(b)) - 1 \quad ,$$

then there exist $L+1$ polynomials in n, $e_0(n), e_1(n), \ldots, e_L(n)$, *not all zero*, and a rational function $R(x,n)$ such that $G(n,x) := R(n,x)F(n,x)$ satisfies

$$\sum_{i=0}^{L} e_i(n) F(n+i,x) = D_x G(x,n) \quad . \qquad (GertDoron)$$

If $F(n, \pm\infty) = 0$ (and hence $G(n, \pm\infty) = 0$) it follows, by integrating from $-\infty$ to ∞ that

$$a(n) := \int_{-\infty}^{\infty} F(n,x)dx \quad ,$$

satisfies the linear recurrence equation with polynomial coefficients

$$\sum_{i=0}^{L} e_i(n) a(n+i) = 0 \quad .$$

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Proof: Let L, for now, be *any* positive integer. Let

$$\overline{H}(n,x) = e^{a(x)/b(x)} \cdot \left(\prod_{p=1}^{P} S_p(x)^{\alpha_p} \right) \cdot \frac{s(x)^n}{t(x)^{n+L}} \quad .$$

We have

$$\sum_{i=0}^{L} e_i(n) F(x, n+i) = h(x) \cdot \overline{H}(x,n) \quad ,$$

where

$$h(x) := \sum_{i=0}^{L} e_i(n) POL(n+i, x) s(x)^i t(x)^{n-i} \quad .$$

Let $q(x)$ and $r(x)$ be the numerator and denominator, respectively, of the logarithmic derivative of $\overline{H}(n,x)$, i.e.

$$\frac{D_x \overline{H}(n,x)}{\overline{H}(n,x)} = \frac{q(x)}{r(x)}$$

Write

$$G(n,x) = \overline{H}(n,x) \cdot r(x) \cdot X(x) \quad , \qquad (Ansatz)$$

where $X(x)$ is a polynomial to be determined. Now $(GertDoron)$ is equivalent to

$$(r'(x) + q(x)) \cdot X(x) + r(x) X'(k) = h(x) \qquad (ContGosper)$$

Let $M := deg(h) - max(deg(r' + q), deg(r) + 1)$, and write $X(x)$ with undetermined coefficients. Plugging this into $(ContGosper)$, and equating coefficients, results in $deg(h) + 1$ equations for $L + M + 2$ unknowns. In order to guarantee a solution, we need

$$L + M + 2 > deg(h) + 1 \quad ,$$

in other words

$$(L + M + 2) - (deg(h) + 2) \geq 0 \quad ,$$

in other words,

$$L \geq max(deg(r' + q), deg(r) + 1) \quad .$$

We leave it to the reader to verify that the expression on the right is

$$L = deg(b) + deg(s) + deg(t) + \sum_{p=1}^{P} deg(S_p) + max(deg(a), deg(b)) - 1 \quad . \square$$

REFERENCES