The (Ordinary) Generating Functions Enumerating 123-Avoiding Words with r occurrences of each of 1,2, ..., n are Always Algebraic

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Abstract: The set of 123-avoiding *permutations* (alias words in $\{1, ..., n\}$ with exactly 1 occurrence of each letter) is famously enumerated by the ubiquitous Catalan numbers, whose generating function $C(x)$ famously satisfies the algebraic equation $C(x) = 1 + xC(x)^2$. Recently, Bill Chen, Alvin Dai, and Robin Zhou found (and very elegantly proved) an algebraic equation satisfied by the generating function enumerating 123-avoiding words with **two** occurrences of each of $\{1, \dots, n\}$. Inspired by the Chen-Dai-Zhou result, we present an algorithm for finding such an algebraic equation for the ordinary generating function enumerating 123-avoiding words with exactly r occurrences of each of $\{1, \dots, n\}$ for **any** positive integer r, thereby proving that they are *algebraic*, and not merely D-finite (a fact that is promised by WZ theory). Our algorithm consists of presenting an *algebraic enumeration scheme*, combined with the Buchberger algorithm.

Introduction

Recall that a word $w = w_1 \dots w_n$ in an ordered alphabet contains a *pattern* σ (a certain permutation of $\{1, ..., k\}$) if there exist

$$1 \leq i_1 < i_2 < \ldots < i_k \leq n$$

such that the subword $w_{i_1} \dots w_{i_k}$ is *order isomorphic* to σ; in other words $w_{i_1}, \dots, w_{i_k}$ are distinct, and if you replace the smallest entry by 1, the second smallest entry by 2, etc., you would get σ.

For example, the word *mathisfun* contains the pattern 132, since (inter alia) the subword *hsn* is order-isomorphic to 132 (under the usual lexicographic order).

A word w avoids the pattern σ if it does not contain it. One is interested in enumerating words, of a given length and given alphabet-size, avoiding one or more patterns.

In a remarkable PhD thesis, under the guidance of guru Herbert S. Wilf, Alexander Burstein ([Bu]) initiated the study of *forbidden patterns* (alias *Wilf classes*) in *words*, extending the very active and fruitful research on *forbidden patterns* in *permutations* initiated by Donald Knuth, Rodica Simion, Richard Stanley, Herbert Wilf, and others. For the current *state of the art* of the latter, see [Wiki]. Burstein's pioneering thesis was extended by quite a few people, and the current knowledge is described in the lucid and insightful research monographs [HM] and [Ki]. A systematic approach for computer-assisted enumeration of words avoiding a given set of patterns, extending the work of Zeilberger and Vatter for permutations (see [Z4] and its references), was initiated by Lara Pudwell ([P]). Some of the recent work (e.g. [GGHP]) is phrased in the equivalent language of *ordered set partitions*. This equivalence is cleverly used in Anisse Kasraoui's ([Ka]) recent article.

Most of this work concerns the set of *all* words avoiding a pattern. In a very interesting recent paper [GGHP], the authors consider (in the equivalent language of ordered set partitions), among other problems, the problem of enumerating 123-avoiding words of length $2n$ where each of the n

letters $\{1,2,\ldots,n\}$ occurs exactly twice, and conjectured a certain second-order linear recurrence equation with polynomial coefficients. They apparently did not realize that, in their case, it was possible to justify it by a (fully rigorous, or at least rigorizable) *hand-waving* argument. By general 'holonomic nonsense' ([Z1]) it is known beforehand that there is *some* such linear recurrence, and it is possible to bound the order, thereby justifying, *a posteriori*, the guessed recurrence, provided that it is checked for sufficiently many initial values. A more direct proof was given by Chen, Dai, and Zhou ([CDZ]), who proved the **stronger** statement that the generating function is *algebraic*, and even found the defining equation explicitly:

$$1-(2\,x+1)\,F^2+x\,(x+4)\,F^4=0 \quad .$$

Using Comtet's algorithm ([Co], see also [S]) for deducing, out of the algebraic equation, a linear differential equation for the generating function, and hence a linear recurrence for the sequence itself, Chen, Dai and Zhou proved the [GGHP] conjecture directly.

In the present article we will generalize this and prove that, for **every** positive integer r, the ordinary generating function enumerating 123-avoiding words of length rn where each of the n letters of $\{1,2,\ldots,n\}$ occurs exactly r times, is algebraic, and present an algorithm for finding the defining equation. Alas, since at the end it uses the memory-heavy, and exponential time, Buchberger's algorithm for finding Gröbner bases, our computer (running Maple) only agreed to explicitly find the next-in-line, the analogous equation for $r=3$:

$$(4\,x+1)^2+\left(64\,x^2+48\,x-1\right)F^2-2\,x\left(128\,x^2+108\,x+27\right)F^4-16\,x^2\,(32\,x+27)\,F^6+x^2\,(32\,x+27)^2\,F^8=0 \quad .$$

This took less than a second, but the case $r=4$ took about an hour.

$$x\,(x+1)^2\left(25\,x^2+65\,x+11\right)^2$$

$$+\left(-1024+461716\,x^2-1521500\,x^6+42500\,x^7-276672\,x^3-7480160\,x^4-6516800\,x^5+49271\,x\right)F^2$$

$$+(223797652\,x^3+1024+628250\,x^7+10000\,x^8-57924600\,x^6$$

$$+827342646\,x^4+1098116930\,x^5+842512\,x+24970546\,x^2)F^4$$

$$+4\,x\left(-2743199\,x-5955159\,x^2-6439500\,x^6+127500\,x^7+28100475\,x^5-108800+58215739\,x^3+187145995\,x^4\right)$$

$$+x(7091445146\,x^3+96468480\,x+1387805641\,x^2-471787725\,x^6+2772000\,x^7$$

$$+60000\,x^8+11351360680\,x^5-458752+15348867846\,x^4)F^8$$

$$+4\,x^2\,(5\,x-256)\,(4\,x+1)\left(25500\,x^5-977800\,x^4+15739435\,x^3+9911721\,x^2+2082455\,x+138496\right)F^{10}$$

$$+2\,x^2\left(256+6503\,x+8658\,x^2+11845\,x^3+200\,x^4\right)(5\,x-256)^2\,(4\,x+1)^2\,F^{12}$$

$$+4\,x^3\,(58+85\,x)\,(5\,x-256)^3\,(4\,x+1)^3\,F^{14}$$

$$+x^3\,(5\,x-256)^4\,(4\,x+1)^4\,F^{16}\;=\;0 \quad .$$

We didn't even try the case $r = 5$.

However, since we know, once again (now even without using Zeilberger's holonomic theory) that the generating function is D-finite, since it has the stronger property of being algebraic, it justifies **rigorously** guessing a linear recurrence equation with polynomial coefficients, which enables one to compute, in *linear time*, any term of the enumerating sequence. We succeeded, using our algorithm, to be described below (which in particular enables a very fast enumeration of many terms of the enumerating sequences), in discovering such recurrences for $1 \leq r \leq 5$, and using [Z2] we (or rather our beloved servant, Shalosh B. Ekhad, running Maple) found precise asymptotics for these cases. This enables us to formulate the following intriguing conjecture, and the second-named author (DZ) is pledging a \$100 donation to the OEIS foundation, in honor of the first prover.

Conjecture: Let $w_r(n)$ be the number of 123-avoiding words of length rn with r occurrences of each of $\{1, \dots, n\}$. Then

$$\lim_{n \to \infty} \frac{w_r(n)}{w_r(n-1)} = (r+1)\, 2^r \qquad .$$

More strongly, $w_r(n)$ is asymptotically $C_r((r+1)2^r)^n \cdot n^{-3/2}$, where C_r is a 'nice' constant (probably $\frac{1}{\sqrt{\pi}}$ times the square-root of a rational number that depends 'nicely' on r).

Using the Maple package `Words123` accompanying this article, we proved it for $r \leq 5$ (but we were unable to guess an expression for C_r in terms of r from the five data points).

Speaking of the OEIS, currently only the cases $r = 1$ (of course!) and $r = 2$ ([OEIS], sequence A220097) are there. We hope to remedy this soon, and enter, at least, $w_r(n)$ for $3 \leq r \leq 5$. For $r = 3$, although it they are not in OEIS, the first ten terms are already in cyberspace (more precisely, in Lara Pudwell's website).

Some Crucial Background and Zeilberger's Beautiful Snappy Proof that 123-Avoiding Words are Equinumerous with 132-Avoiding Words

Burstein [B] proved that the number of *all* words in a given (ordered) alphabet of a given length n avoiding 123 is the same as the number of words avoiding 132, and hence, via trivial symmetry, all patterns of length 3 have the same enumeration. The stronger result that this is still true if one specifies the number of occurrences of each letter was first proved in [AAAHH], but the *proof from the book* appeared in the half-page *gem*, [Z3]. Since this lovely proof deserves to be better known, we reproduce it here.

Define a mapping F on a word w in the alphabet $\{1, 2, \dots, n\}$ recursively as follows. If w is empty, then $F(w) := w$. Otherwise, $i := w_1$, and let W be the word obtained from w by first beheading it, and then replacing all letters larger than $i+1$ by $i+1$, and let s be the sub-sequence of w obtained by deleting the letters $\leq i$. Let $\bar{s}$ be the reverse of s. Let $V := F(W)$, and let U be the word obtained from V by replacing (in order) the letters that are $i+1$ by the members of $\bar{s}$. Finally let $F(w) := iU$.

F is an involution that sends 123-avoiding words to 132-avoiding ones, and vice versa. This follows from the fact that s above is non-increasing and non-decreasing respectively. Hence, for any vector of non-negative integers $(a_1, \ldots, a_n)$ amongst the $(a_1 + \ldots + a_n)!/(a_1! \cdots a_n!)$ words with a_1 1's, $\ldots$, a_n n's, the number of those that avoid the pattern 123 equals the number of those that avoid 132,

It also follows that we have a quick recurrence that enables us to compute the number of such words, which we will call $A(a_1, \ldots, a_n)$:

$$A(a_1, \ldots, a_n) = \sum_{i=1}^{n} A(a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1} + \ldots + a_n) \quad .$$

Another important consequence (which also follows from the Robinson-Schenstead-Knuth algorithm) is that $A(a_1, \ldots, a_n)$ is *symmetric* in its arguments.

Because of the equinumeracy of all patterns of length 3, we can consider 231-avoiding words.

Important Definitions

Let $\mathcal{W}_r(n)$ be the set of 231-avoiding words in the alphabet $\{1, \ldots, n\}$ with exactly r occurrences of each letter.

Also, let $w_r(n)$ be the number of elements of $\mathcal{W}_r(n)$.

Define the 'global set'

$$\mathcal{W}_r := \bigcup_{n=0}^{\infty} \mathcal{W}_r(n) \quad .$$

Let $g_r(x)$ be its *weight enumerator* with respect to the weight $w \to x^{length(w)}$. Note that $g_r(x) = f_r(x^r)$, where $f_r(x)$ is the generating function of the sequence $w_r(n)$,

$$f_r(x) := \sum_{n=0}^{\infty} w_r(n) x^n \quad .$$

We will soon show how, for any *specific*, given, positive integer r, to obtain an algebraic equation (i.e. a polynomial $P_r(x, F)$ with integer coefficients such that $P_r(x, f_r(x)) = 0$), but let's srart with some *warm-ups*.

First Warm-Up: r=1

$\mathcal{W}_1$ is the set of *all* permutations (of any length!) that avoid the pattern 231. Let the weight of a permutation π be $x^{length(\pi)}$. Consider any member π of that set. It may happen to be the empty permutation, of course (weight 1), or else it has a largest element; let's call that element n. All the entries to the left of n must be **smaller** than all the elements to the right of n (or else a 231 pattern would emerge), and each portion must be 231-avoiding in its own right. If the location of n is at the i-th place, then the portion to the left of n is a 231-avoiding permutation of $\{1,\dots,i-1\}$ and the portion to the right is a 231-avoiding permutation of $\{i,\dots,n-1\}$. Conversely, if π_1 and π_2 are 231-avoiding permutations of $\{1,\dots,i-1\}$ and $\{i,\dots,n-1\}$ respectively, then $\pi_1 n \pi_2$ is a 231-avoiding permutation of length n, since no trouble can arise by joining them. Hence,

$$f_1(x) = 1 + x f_1(x)^2 \quad ,$$

giving the good-old Catalan numbers.

Second Warm-Up: r=2

The following argument is inspired by the beautiful proof in [CDZ], but is phrased in such a way that will make it transparent how to generalize it for general r.

Let $g(x)$ be the weight-enumerator of $\mathcal{W}_2$. Recall that $\mathcal{W}_2$ is the set of all 231-avoiding words whose letters consist of $\{1,1,\dots,n,n\}$ for some $n \geq 0$, and the weight is $x^{length(w)} = x^{2n}$).

(Note that $g(x) = f_2(x^2)$), so once we have $g(x)$ we will have $f_2(x)$ immediately.)

Consider a typical member of $\mathcal{W}_2$, and let n be its largest element (i.e. it is of length $2n$). Let i be the location of the **leftmost** occurrence of n. Notice, just as before, that the entries to the left of that first n must be $\leq$ the entries to the right of that n, and each portion is 231-avoiding in its own right, and conversely, if you place such 231-avoiding words with these entries to the left and right of that leftmost n, you will not cause any trouble, and get a 231-avoiding word whose entries are $\{1,1,2,2,\dots,n,n\}$.

Case I: i is odd, i.e. $i = 2j+1$.

Then the entries to the left of that first n are $\{1,1,\dots,j,j\}$ and the entries to the right are $\{j+1,j+1,\dots,n-1,n-1,n\}$. The generating function of the left part is our $g(x)$, but the entries to the right are a new combinatorial creature: a 231-avoiding word with all the letters occurring twice, except for one of them (which by symmetry can be taken to be '1') that only occurs once. So let's give the set $\mathcal{W}_2$ the new name $\mathcal{W}_2^{(0,0)}$, and let $\mathcal{W}_2^{(1,0)}$ be the union of the sets of 231-avoiding words on $\{1,2,2,3,3,\dots,n,n\}$, for all $n \geq 0$. Let $g^{(1,0)}(x)$ be its weight-enumerator. Hence the total weight-enumerator of Case I is

$$x\, g^{(0,0)}(x)\, g^{(1,0)}(x) \quad .$$

(The x in front corresponds to the first n separating the two parts).

We will deal with $g_2^{(1,0)}(x)$ in due course, but now let's proceed to Case II.

Case II: i is even, i.e. $i = 2j$.

Once again let its length be $2n$ (so the largest entry is n). The entries to the left of that first n are $\{1,1,\dots,j-1,j-1,j\}$, and the entries to the right are $\{j,j+1,j+1,\dots,n\}$. The generating function of the left part is the already familiar $g^{(1,0)}(x)$, but the right part is a new combinatorial creature; namely, a 231-avoiding word with all the letters occurring twice, **except** for *two* of them (that by symmetry may be taken to be the smallest and the largest) that only occur *once*. Let's call this set $\mathcal{W}_2^{(1,1)}$, and its weight-enumerator $g^{(1,1)}(x)$. Hence the total weight of Case II is $x\,g^{(1,0)}(x)\,g^{(1,1)}(x)$.

Combining the two cases, plus the empty permutation, leads to the following equation

$$g^{(0,0)}(x) = 1 \,+\, x\,g^{(0,0)}(x)\,g^{(1,0)}(x) \,+\, xg^{(1,0)}(x)\,g^{(1,1)}(x) \quad . \qquad (Eq00)$$

We have two new *uninvited guests*, $g^{(1,0)}(x)$ and $g^{(1,1)}(x)$. Using the same reasoning as above, the readers are welcome to convince themselves that

$$g^{(1,0)}(x) \;=\; xg^{(0,0)}(x)^2 \,+\, xg^{(1,0)}(x)^2 \quad , \qquad (Eq10)$$

$$g^{(1,1)}(x) \;=\; x\,g^{(0,0)}(x)\,g^{(1,0)}(x) \,+\, x\,g^{(1,0)}(x)\,(1+g^{(1,1)}(x)) \quad . \qquad (Eq11)$$

Solving this *algebraic scheme*, a system of three algebraic equations $\{Eq00, Eq10, Eq11\}$ in the three '*unknowns*' $\{g^{(0,0)}(x)\,,\, g^{(1,0)}(x)\,,\, g^{(1,1)}(x)\}$, using Gröbner bases (or, in this simple case it could be easily done by hand) gives an algebraic equation satisfied by $g^{(0,0)}(x)$, and hence, after replacing x^2 by x, the [CDZ] equation for $f_2(x)$ mentioned above:

$$1-(2\,x+1)\,f_2(x)^2 + x\,(x+4)\,f_2(x)^4 = 0 \quad .$$

The General Case

For $0 \le i \le j \le r-1$ and $n \ge 0$, let $\mathcal{W}_r^{(i,j)}(n)$ be the set of 231-avoiding words of length $rn+i+j$, in the alphabet $\{1,2,\dots,n,n+1,n+2\}$, with i occurrences of the letter '1', j occurrences of '$n+2$', and exactly r occurrences of the other n letters (i.e. $2,3,\dots,n+1$), and let $\mathcal{W}_r^{(i,j)}$ be the union of $\mathcal{W}_r^{(i,j)}(n)$ over all $n \ge 0$.

By symmetry they have the same weight-enumerator if *any* two letters have i and j occurrences respectively, and the remaining letters each occur exactly r times.

Using the same logic as above, we have the following $\binom{r+1}{2}$ equations, for $0 \le i \le j \le r-1$, where below we make the convention that if $r > s$ then $g^{(r,s)} = g^{(s,r)}$.

$$g^{(i,j)}(x) = \delta_{i,0}\delta_{j,0} \,+\, x\sum_{t=0}^{r-1} g^{(i,t)}(x)\,g^{((r-t)\,mod\,r\,,\,(j-1)\,mod\,r)}(x) \,+\, \sum_{m=0}^{i-1} x^{m+1}\,g^{(i-m\,,\,j-1)}(x) \quad .$$

By *eliminating* $g^{(0,0)}(x)$, and replacing x^r by x, we get the equation of our object of desire $f_r(x)$. In fact, this equation would have several solutions, and the right one is picked by plugging in the first few terms.

Guessing Linear Recurrences for our sequences

Now that we know, even without WZ-theory, that for every positive integer r, the generating function $f_r(x)$ is D-finite, since it has the much stronger property of being algebraic, we immediately know that the sequence itself, $\{w_r(n)\}$, is P-recursive in the sense of Stanley[S]; in other words, it satisfies *some* homogeneous linear recurrence equation with *polynomial* coefficients.

With a very large computer, one should be able to get the algebraic equation for quite a few r, and then use Comtet's algorithm (built-in in the Maple package `gfun`, procedure `algeqtodiffeq` followed by procedure `diffeqtorec`), to get a rigorously derived recurrence. Alas, because our system has $(r+1)r/2$ algebraic equations, and Gröbner bases are notoriously slow, we were only able to do two new cases explicitly, namely $r=3$ and $r=4$, mentioned above. But now that we know *for sure* that such recurrences exist, and it is easy to find a priori bounds for the order, it is easy to justify these empirically-derived recurrences, a posteriori.

But in order to guess complicated linear recurrences, one needs lots of data. Our algebraic scheme implies very fast *nonlinear* recurrences for the coefficients of $g^{(i,j)}(x)$, and in particular for $g^{(0,0)}(x)$, our primary interest. These turn out to be much faster than the 'vanilla' linear recurrence for $A(a_1,\ldots,a_n)$ mentioned above.

The Maple package Words123

Everything (and more!) is implemented in the Maple package `Words123`, available directly from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Words123` ,

or via the front of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/words123.html` ,

that also contains some sample input and output files.

The recurrences for $1 \le r \le 3$

For $r=1$ we get the good-old Catalan numbers

$$-2\,\frac{(1+2\,n)\,w_1\,(n)}{n+2}+w_1\,(n+1)=0 \quad .$$

For $r=2$ we get a new proof of the [GGHP] conjecture (first proved in [CDZ])

$$-3\,\frac{(7\,n+12)\,(1+2\,n)\,(1+n)\,w_2\,(n)}{(2\,n+5)\,(7\,n+5)\,(n+2)}-\frac{\left(528+1426\,n+1215\,n^2+329\,n^3\right)w_2\,(n+1)}{2\,(2\,n+5)\,(7\,n+5)\,(n+2)}+w_2\,(n+2)=0 \quad .$$

For $r=3$ we get

$$-\frac{64}{3}\,\frac{(4\,n+1)\,(2\,n+3)\,(4\,n+3)\,(14\,n+25)\,(n+1)\,w_3\,(n)}{(3\,n+5)\,(1+2\,n)\,(3\,n+7)\,(14\,n+11)\,(n+2)}$$

$$-\frac{8}{3}\cdot\frac{\left(3975+20322\,n+39676\,n^2+37144\,n^3+16736\,n^4+2912\,n^5\right)w_3\,(n+1)}{(3\,n+5)\,(1+2\,n)\,(3\,n+7)\,(14\,n+11)\,(n+2)}+w_3\,(n+2)=0 \quad .$$

See the output file

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oWords123c`

for the recurrences for $w_4(n)$ and $w_5(n)$.

The Asymptotics for $1 \leq r \leq 5$

$$w_1(n) \;=\; \frac{1}{\sqrt{\pi}}\cdot 4^n \;\cdot n^{-\frac{3}{2}}\left(1-\frac{9}{8}\,n^{-1}+\frac{145}{128}\,n^{-2}-\frac{1155}{1024}\,n^{-3}\;+\,O(n^{-4})\right) \quad ,$$

$$w_2(n) \;=\; \frac{1}{\sqrt{\pi}}\cdot\frac{3\sqrt{3}}{7\sqrt{7}}\cdot 12^n \;\cdot n^{-\frac{3}{2}}\left(1-\frac{249}{392}\,n^{-1}+\frac{13255}{43904}\,n^{-2}-\frac{2674485}{17210368}\,n^{-3}\;+\,O(n^{-4})\right) \quad ,$$

$$w_3(n) \;=\; \frac{1}{8\sqrt{\pi}}\cdot 32^n \;\cdot n^{-\frac{3}{2}}\left(1-\frac{33}{64}\,n^{-1}+\frac{1105}{8192}\,n^{-2}-\frac{27195}{524288}\,n^{-3}\;+\,O(n^{-4}\right) \quad ,$$

$$w_4(n) \;=\; \frac{1}{6\sqrt{6}\sqrt{\pi}}\;\cdot 80^n \;\cdot n^{-\frac{3}{2}}\left(1-\frac{23}{48}\,n^{-1}+\frac{1621}{23040}\,n^{-2}-\frac{339199}{16588800}\,n^{-3}\;+\,O(n^{-4})\right) \quad ,$$

$$w_5(n) \;=\; \frac{1}{\sqrt{\pi}}\cdot\frac{3\sqrt{3}}{125}\cdot 192^n\cdot n^{-\frac{3}{2}}\left(1-\frac{471}{1000}\,n^{-1}+\frac{389141}{10000000}\,n^{-2}-\frac{162387477}{50000000000}\,n^{-3}\;+\,O(n^{-4})\right) \quad .$$

Warning: the above asymptotic expressions are fully rigorous except for the constants in front, which are only conjectured.

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Nov. 18, 2014