**Experimenting with Standard Young Tableaux**

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*In fond memory of Herbert Wilf and Albert Nijenhuis, and in honor of Curtis Greene*

**Abstract**: Using Symbolic Computation with Maple, we can discover lots of (rigorously-proved!) facts about Standard Young Tableaux, in particular the distribution of the entries in any specific cell, and the sorting probabilities.

**Maple package**

This article is accompanied by a Maple package, `SYT.txt`, available from:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/SYT.txt` .

The web-page of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/syt.html` ,

contains many input and output files, some of which will be referred to later.

**Preface**

One of the most *iconic* objects in mathematics, both *concrete* [K], and *abstract* [F], are Standard Young Tableaux [Wi]. Recall that an *integer partition*, or *partition*, for short, aka *shape*, of a non-negative integer $n$, with $k$ **parts**, is a non-increasing sequence of **positive** integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),$$

where

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0,$$

such that

$$\lambda_1 + \ldots + \lambda_k \,=\, n.$$

The **Ferrers diagram** (or **Young diagram**) of a partition $\lambda$ is a left-justified array of dots (or empty boxes) where the top row has $\lambda_1$ dots (boxes), the second row has $\lambda_2$ boxes, ..., the $k$-th row has $\lambda_k$ dots. For example, the Ferrers diagram of $(4, 4, 3, 1)$ is

$$\begin{matrix} * & * & * & * \\ * & * & * & * \\ * & * & * & \\ * & & & \end{matrix} \; .$$

Given a shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $n$ boxes, a **standard Young tableau** is a way of filling the boxes with the integers $\{1, \ldots, n\}$, such that each of them shows up (necessarily once) and both rows and columns are increasing. More formally, it is an array

$$T_{i,j}, \quad \text{where} \quad 1 \leq i \leq k, 1 \leq j \leq \lambda_i,$$

such that $T_{i,j} < T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ whenever they exist.

Here are the five standard Young tableaux of shape $(2, 2, 1)$:

$$\begin{array}{ll} 1 & 2 \\ 3 & 4 \\ 5 & \end{array}, \quad \begin{array}{ll} 1 & 2 \\ 3 & 5 \\ 4 & \end{array}, \quad \begin{array}{ll} 1 & 3 \\ 2 & 4 \\ 5 & \end{array}, \quad \begin{array}{ll} 1 & 3 \\ 2 & 5 \\ 4 & \end{array}, \quad \begin{array}{ll} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}.$$

To see the set of standard Young tableaux of shape `L` in our Maple package, type `SYT(L);` . For example to see the above five tableaux type `SYT([2,2,1]);` .

The total number of standard Young tableaux of shape $\lambda$, denoted by $f^\lambda$, is famously given by the *hook length formula*, or equivalently (and more convenient for us) by the **Young-Frobenius formula** (see [K]).

$$f^\lambda = \frac{(\lambda_1 + \lambda_2 + \ldots + \lambda_k)!}{(\lambda_1 + k - 1)!(\lambda_2 + k - 2)! \cdots \lambda_k!} \cdot \prod_{i=1}^{k} \prod_{j=i+1}^{k} (\lambda_i - \lambda_j + j - i).$$

Fix a shape $\lambda$ and fix a cell $[i, j]$, $1 \leq i \leq k, 1 \leq j \leq \lambda_i$.

Who can be the occupant of that cell?

Calling that occupant $r$, we have:

$$ij \leq r \leq \lambda_1 + \lambda_2 + \ldots + \lambda_{i-1} + \lambda'_1 + \lambda'_2 + \ldots + \lambda'_{j-1} - (i-1)(j-1) + 1,$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_{\lambda_1})$ is the **conjugate partition**.

This gives a certain **probability distribution**. What is it?

For example, with the shape $\lambda = (2, 2, 1)$ again, and the cell $[2, 1]$, the set of possible occupants is $\{2, 3\}$, and the probability of it being 2 is $\frac{3}{5}$ and of it being 3 is $\frac{2}{5}$.

Later on we will be interested not in *specific* shapes, but in general (mostly rectangular) shapes, with a fixed number of rows, but arbitrary (i.e. *symbolic*) shape. Fixing the number of rows to be $k$ (where $k$ is *numeric*), and regarding the shape $(n, n, \ldots, n)$, where $n$ is repeated $k$ times, we would be interested in deriving closed-form expressions (as rational functions of $n$), for the probability distribution of the possible occupants of a given first row cell $[1, j]$, for any given **numeric** integer $j$. Note that the possible occupants of $[1, j]$ are $j, j+1, j+2, \ldots, k(j-1)+1$.

Once we found these expressions in $n$, we can ask about the *limiting distribution*, that Maple can find for us. Then we can also hope to see how it varies with $j$ and look at the *meta-limiting* behavior as $j$ gets larger.

Another kind of question, inspired by the beautiful work of Chan, Pak, and Panova [CPP1, CPP2], is to study the **sorting probabilities**. Given two cells $c_1$ and $c_2$, draw a standard Young tableau *uniformly at random*. Who is bigger?

The occupant of $c_1$ or the occupant of $c_2$?

Of course if the two cells are *related*, i.e. one of them, say $c_2$, is (weakly) below and (weakly) to the right of the other, $c_1$, i.e. in the underlying poset $c_1 < c_2$, then of course, *always* $T_{c_1} < T_{c_2}$, but what if they are **not** related i.e. writing

$$c_1 = [i_1, j_1], c_2 = [i_2, j_2],$$

we have $i_1 < i_2$ but $j_1 > j_2$.

The **sorting probability** is defined by:

$$SP(\lambda, c_1, c_2) := Pr(T_{c_1} > T_{c_2}) - Pr(T_{c_2} > T_{c_1}) = 2\,Pr(T_{c_1} > T_{c_2}) - 1,$$

where $T$ is a random standard Young tableau of shape $\lambda$.

In particular, following Chan-Pak-Panova, we are interested in the **minimal** (absolute value) of the sorting probabilities, over all pairs of cells, as the shapes get larger.

Going back to the shape $\lambda = (2, 2, 1)$, we see that for the first two tableaux the occupant of $(1, 2)$ is less than the occupant of $(2, 1)$, while for the last three ones it is the reverse. Hence the probability of $T_{1,2} < T_{2,1}$ is $\frac{2}{5}$, and so $SP(221, [1, 2], [2, 1]) = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$. The **minimal sorting probability** for that shape is also $\frac{1}{5}$ (check!).

**Simulation**

One way to answer these questions, *approximately*, is via *simulation*. The amazing Greene-Nijenhuis-Wilf [GrNW] algorithm (that also led to a beautiful probabilistic proof of the *hook length formula*) inputs a shape, $\lambda$, and outputs, *uniformly at random*, one of the $f^\lambda$ standard Young tableaux of that shape. By sampling many of them, we can get approximations to the quantities of interest.

Procedure `GNW(L)` in our Maple package `SYT.txt` implements the Greene-Nijenhuis-Wilf algorithm. For example, try `GNW([4,3,2]);` , in order to get, uniformly at random, one of the 168 standard Young tableaux of shape $(4, 3, 2)$.

To get approximations for the probability generating function, using the variable `x`, of the distribution of the occupants of cell `c` in a random Young tableau of shape `L`, by sampling `K` random tableaux, type `SiOcGF(L,c,x,K);`.

For example for the shape $(4,4,4)$ and the cell $[2,2]$, with 10000 tries, type:

`SiOcGF([4,4,4],[2,2],x,10000);`

getting something like (of course it is slightly different each time)

$$.1090000000\,x^7 + .2875000000\,x^6 + .3639000000\,x^5 + .2396000000\,x^4.$$

To get approximations for the sorting probability of cell `c1` vs. cell `c2`, in the shape `L`, by sampling $K$ tableaux: enter: `SiPr(L,c1,c2,K);`. For example, `SiPr([3,3,3],[1,2],[2,1],10000);` would give something like `0.010400000`. Of course, in this particular case the exact answer is obviously zero, by symmetry, so getting something close to 0 is a good *sanity check*.

**Symbol crunching in order to find the probability distribution of the occupants of a specific cell in a symbolic shape**

For the sake of exposition, we will mostly be concerned with standard Young tableaux of rectangular shape. Fix the number of rows $k$ (implicitly taken $\leq n$, and the case $k > n$ can be handled by the symmetries), and consider the shape

$$(n, n, \ldots, n),$$

where $n$ is repeated $k$ times. More generally all our algorithms carry over to the general *symbolic* shape

$$(n_1, n_2, \ldots, n_k),$$

where $n_1 \geq n_2 \geq n_k \geq 0$, and they are all left symbolic. Using Young-Frobenius we get an **explicit** formula for their total number in the form of a certain **rational function**, of $(n_1, \ldots, n_k)$ times the **multinomial coefficient**

$$\frac{(n_1 + \ldots + n_k)!}{n_1! \cdots n_k!}.$$

In the special case of a $k \times n$ rectangular shape: (i.e. $(n, \ldots, n)$, where $n$ is repeated $k$ times), it is a certain rational function of $n$ (namely $\frac{(k-1)!}{(n+1)_1 (n+1)_2 \cdots (n+1)_{k-1}}$, where, as usual $(x)_m := x(x+1)\cdots(x+m-1)$) times

$$\frac{(n\,k)!}{n!^k}.$$

Now also fix a specific (*numeric*) cell, $c = [i,j]$, and a specific (numeric) integer $r$. We want an **explicit** formula, as a **rational function** of $n$, for the probability that, when you draw (say using the GNW algorithm) *uniformly at random*, one of the $\frac{(n\,k)!(k-1)!}{n! \cdots (n+k-1)!}$ standard Young tableaux, of shape $n^k$, that the occupant of the cell $c = [i,j]$ is the integer $r$, in symbols: $Pr(T_{ij} = r)$.

We will soon explain how to do it, but you are welcome to try it out first using our Maple package `SYT.txt`. Let's give a few examples.

- To get the explicit expression for the probability that the occupant of cell $[1,3]$ in a random standard Young tableau of shape $(n,n,n)$ happens to be 7, type: `OcCs([n,n,n],7,[1,3]);` getting

$$\frac{5n\,(n+1)^2\,(n+2)}{9\,(3n-1)\,(3n-2)\,(3n-4)\,(3n-5)}.$$

• For a more complicated example, to get the expression for the probability that the occupant of cell $[3,3]$ in a random standard Young tableau of shape $(n,n,n)$ happens to be 13, type: `OcCs([n,n,n],13,[3,3]);` getting

$$\frac{110n^2\,(-1+n)\,(n+1)^2\,(2+n)\left(233n^2-1933n+3984\right)}{81\,(-1+3n)\,(-2+3n)\,(-4+3n)\,(-5+3n)\,(-7+3n)\,(-8+3n)\,(-10+3n)\,(-11+3n)}.$$

• For yet another example, `OcCs([n[1],n[2],n[3]],3,[1,2]);` gives the rational function (in $n_1,n_2,n_3$) for the probability that cell $[1,2]$ would be occupied by 3 for the three-rowed shape $(n_1,n_2,n_3)$:

$$\frac{n_1^2n_2+n_1^2n_3+n_1n_2^2+2n_1n_2n_3+n_1n_3^2+n_2^2n_3+n_2n_3^2-n_1n_2-n_1n_3+n_2^2-n_3n_2+2n_3^2-2n_2-6n_3}{(n_1-2+n_2+n_3)\,(n_1-1+n_2+n_3)\,(n_1+n_2+n_3)}.$$

• If the cell is at the first row, $c=[1,j]$, for some $j>1$, then there are only finitely many possible occupants $r$, namely $r=j,j+1,\ldots,k(j-1)+1$, and to get the probability generating function, using the variable $x$, type `OcGFs1(L,j,x);`.

For example, entering `OcGFs1([n,n,n],2,x);` gives you

$$\frac{2\,(-1+n)\,x^2}{-1+3n}+\frac{8\,(-1+n)\,(n+1)\,x^3}{3\,(-1+3n)\,(-2+3n)}+\frac{(n+1)\,(2+n)\,x^4}{3\,(-1+3n)\,(-2+3n)},$$

meaning that the cell $[1,2]$ in a standard Young tableau of shape $(n,n,n)$ is occupied by either 2, 3, or 4, with respective probabilities of $\frac{2(-1+n)}{-1+3n}$, $\frac{8(-1+n)(n+1)}{3(-1+3n)(-2+3n)}$, and $\frac{(n+1)(2+n)}{3(-1+3n)(-2+3n)}$.

To get the limiting distribution as $n\to\infty$, as well as the expectation, variance, and the first few moments up to the $K$-th, try `OcGFs1L(L,n,i,x,K);` .

**How does Maple perform these amazing calculations? In other words how does it compute $Pr(T_{ij}=r)$ for a random standard Young tableau of a symbolic shape?**

Given a (symbolic, or numeric) shape $\lambda$, a cell $c=[i,j]$, and an integer $r$, how can it happen that $T_{ij}=r$? The cells occupied by $\{1,2,\ldots,r\}$ form a certain standard Young tableau with $r$ cells, that is a certain subshape, which must contain the cell $c=[i,j]$, that must be a corner, of course. So let's ask our beloved computer to find all the shapes with $r$ cells that contain the cell $[i,j]$ as a corner, or equivalently the set of partitions, $\nu$, of $r$ with at least $i$ rows such that $\nu_i=j$. Let's call this (finite) set $S([i,j],r)$.

Let, as usual, $f^{\lambda/\nu}$ denote the number of standard Young tableau of **skew-shape** $\lambda/\nu$ (recall that these are shapes where $\nu$ is a subshape of $\lambda$, and the cells of $\nu$ are removed). Then our quantity of interest is

$$\sum_{\nu \in S([i,j];r)} f^{\nu'} f^{\lambda/\nu},$$

where $\nu'$ is the shape $\nu$ with the cell $[i,j]$ removed.

The **number** $f^{\nu'}$ is easily computed using the Young-Frobenius formula. How do we compute the (symbolic) **expression** $f^{\lambda/\nu}$?

Recall that standard Young tableaux of shape $\lambda = (\lambda_1, \dots, \lambda_k)$ are in bijection with **walks** from the **origin** to the point $\lambda$ in the $k$-dimensional discrete lattice $\mathbf{N}^k$, that always must stay in the region

$$x_1 \geq x_2 \geq \ldots \geq x_k \geq 0.$$

Similarly, standard Young tableaux of *skew-shape* $\lambda/\nu$ are in bijection with such 'sub-diagonal' walks from $\nu$ to $\lambda$. Now following the ideas of André [Z] (see also [GeZ]), put mirrors on the hyperplanes

$$x_1 - x_2 = -1, x_2 - x_3 = -1, \dots, x_{k-1} - x_k = -1,$$

and look at the set of $k!$ images of the point $\nu$ under the action of the group generated by these $k-1$ reflections. As is well-known (and fairly easy to see), the underlying group is the symmetric group $S_k$, and the sign is 1 or $-1$ according to whether the number of inversions is even or odd. Calling the set of images $IMAGE(\nu)$, we have:

$$f^{\lambda/\nu} = \sum_{\mu \in IMAGE(\nu)} \pm W(\mu, \lambda),$$

where $W(\mu,\lambda)$ is the **number** of walks in the lattice from $\mu$ to $\lambda$ given by the **multinomial coefficient**

$$\frac{(\lambda_1 + \ldots + \lambda_k - \mu_1 - \ldots - \mu_k)!}{(\lambda_1 - \mu_1)!(\lambda_2 - \mu_2)! \cdots (\lambda_k - \mu_k)!}.$$

But since we are interested in probabilities, we can divide everything by $f^\lambda$ and stay in the realm of rational functions. This is implemented in the procedure `Swee(L,M)`.

**Computing the sorting probabilities for symbolic shape and any two cells where one of them is at the first row**

The **numeric** procedure `Pr(L,c1,c2)`, for a random standard Young tableau of shape `L`, manually finds the sorting probability of cell `c1` vs. cell `c2`, and the numeric procedure `MinPr(L)` finds the minimal sorting probabilities among all pair of cells, followed by the 'champions'. For example, if you type `MinPr([10,4,3]);`, you would get

`1/273, {[[1, 5], [3, 1]]},`

meaning that the minimum (absolute value) of the sorting probabilities among all the $\binom{17}{2} = 136$ pairs of cells is $\frac{1}{273}$ and it is achieved with the pair of cells $[1,5]$ and $[3,1]$. But we want to do things **symbolically**. Alas, things get complicated if neither cells are at the first row.

But we can, exactly, and **symbolically**, compute a closed-form expression, as a rational function of the symbols, of the sorting probabilities between any cell $[1, j]$ on the first row and any cell below it (to the left, of course, or else the sorting probability is trivially $-1$).

This is implemented in procedure `PrS(L,j,c2)`, where `L` is the symbolic shape and `c2` is the cell below the first row that we compare it to `[1,j]`. For example, to get the sorting probability of cell $[1, 3]$ vs. the cell $[2, 2]$, for the shape $(n, n, n)$, type

`PrS([n,n,n],3,[2,2]);` getting

$$-\frac{(17n-4)(n-3)}{3(3n-1)(3n-4)}.$$

**How does Maple find the symbolic sorting probabilities?**

How do we do it? Look at all the possible occupants of cell $c_1 = [1, j]$ (there are finitely many of them). Suppose it happens to be $r$. How can it be larger than the occupant of cell $c_2 = [m_1, m_2]$?. We find the (finite) set of shapes with $r$ cells that include $c_1 = [1, j]$, and in addition it is a corner. In other words all the shapes $\nu$ with $r$ cells such that $\nu_1 = j$, $\nu$ has at least $m_1$ rows, and $m_2 \leq \nu_{m_1}$.

As before add-up $f^{\nu'}$ times $f^{\lambda/\nu}$, and then add-them-up for all possible legal occupants of $[1, j]$. Getting a nice (or not so nice, but still explicit) expression for $Pr(T_{1,j} > T_{m_1,m_2})$, and hence for the sorting probability $2\,Pr(T_{1,j} > T_{m_1,m_2}) - 1$ .

Of course, we always divide by $f^{\lambda}$ (but this is already built-in in all our *macros*).

**A one-line proof that the minimal sorting probability for the Catalan poset is $O\left(\frac{1}{n}\right)$**

In a deep and beautiful work [CPP2], the authors proved that the minimal sorting probability of the Young lattice, as the shapes get larger, tends to 0. In the more specific paper [CPP1], they proved, by an ingenious and delicate asymptotic analysis, that for the two-rowed case, $[n, n]$, (what they call the *Catalan poset*), it is $O\left(\frac{1}{n^{5/4}}\right)$. But using our Maple package, we can get, *without human effort*, a (rigorous!) proof that it is at least $O\left(\frac{1}{n}\right)$.

Indeed, entering in our Maple package `SYT.txt`, the command: `PrS([n,n],3,[2,1]);`, we get, in one *nano-second* :

$$\frac{3}{2n-1}.$$

So we have the following computer-generated proposition (that admittedly could have been done by humans only using paper and pencil).

**Proposition**: The sorting probability of the cell $[1, 3]$ and the cell $[2, 1]$ in a random standard Young tableau of shape $(n, n)$ is

$$\frac{3}{2n-1} = \frac{3}{2}\cdot\frac{1}{n} + \frac{3}{4}\cdot\frac{1}{n^2} + \frac{3}{8}\cdot\frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

Hence the minimal sorting probability of the Catalan lattice is $O\left(\frac{1}{n}\right)$.

Similarly, typing `PrS([n,n],5,[2,2]);` gives the following proposition.

**Proposition**: The sorting probability of the cell $[1,5]$ and the cell $[2,2]$ in a random standard Young tableau of shape $(n,n)$ is

$$\frac{45n^2-135n+30}{2\,(2n-5)\,(2n-1)\,(2n-3)} = \frac{45}{16}\cdot\frac{1}{n}+\frac{135}{32}\cdot\frac{1}{n^2}+\frac{75}{16}\cdot\frac{1}{n^3}+O\left(\frac{1}{n^4}\right).$$

Hence, again, the minimal sorting probability of the Catalan poset is $O\left(\frac{1}{n}\right)$.

Procedure `FindZerO(L,n,K)` searches for all pairs of cells $c_1=[1,j], c_2=[m_1,m_2]$ where $c_1$ is in the first row, and $j, m_2 \leq K$, such that the sorting probability tends to 0 (and hence is, of course $O(1/n)$). Alas, except for the above two pairs (for the Catalan poset), none exists for $K=100$. Note that here we really lucked out, since the pairs $\{[1,3],[2,1]\}$ and $\{[1,5],[2,2]\}$ are *numeric* (and small), and given an upper bound for the minimal sorting probability. In order to get to the *true* minimum, **both** $c_1$ and $c_2$ must be *symbolic* (that what was essentially done in [CPP1] and [CPP2] with great human effort).

**The special case of the Catalan poset (2-rowed tableaux)**

For the Catalan case things can get much faster (as noticed in [CPP1]) and the procedures implementing this can be found by typing `ezraD();` .

`Anij(n,i,j)` is a faster version of PrS([n,n],i,[2,j]). It turns out that in this case we can get closed-form expressions, for the occupancy distribution of an arbitrary cell $[1,i]$ at the first row of a standard Young tableau of shape $(n,n)$ for **symbolic** $i$, that entail, in turn, closed-form expressions for the *limiting distribution* as $n$ goes to $\infty$, and **surprise!** we can get explicit expressions for the average, variance, and higher moments for that limiting distribution for symbolic $i$, and even the *meta-limiting* behavior, as $i$ goes to infinity.

We have

**Proposition**: The expectation of the occupant of cell $[1,i]$ in a random standard Young tableau of shape $(n,n)$, as $n$ goes to infinity is

$$2i+2-\frac{2\cdot 4^{-i}\,(1+2i)!}{i!^2},$$

confirming Richard Stanley's observation mentioned in [CPP1], Eq. (5.1). The asymptotics is

$$2i+2-\frac{4}{\sqrt{\pi}}i^{1/2}-\frac{3}{2\sqrt{\pi}}i^{-1/2}+\frac{7}{32\sqrt{\pi}}i^{-3/2}-\frac{9}{256\sqrt{\pi}}\,i^{-5/2}+O\left(i^{-7/2}\right).$$

The variance is

$$-\frac{4\cdot 16^{-i}\,(1+2i)!^2}{i!^4}-\frac{2\cdot 4^{-i}\,(1+2i)!}{i!^2}+6i+6.$$

The limiting (as $i$ goes to infinity) *skewness* is $\frac{2(5\pi-16)\sqrt{2}}{(3\pi-8)^{\frac{3}{2}}} = -0.4856928234\ldots$

The limiting (as $i$ goes to infinity) *kurtosis* is $\frac{15\pi^2+16\pi-192}{(3\pi-8)^2} = 3.108163850\ldots$

The limiting (as $i$ goes to infinity) *scaled-fifth-moment* is $\frac{2\left(51\pi^2-80\pi-256\right)\sqrt{2}}{(3\pi-8)^{\frac{5}{2}}} = -4.642979574\ldots$

The limiting (as $i$ goes to infinity) *scaled-sixth-moment* is $\frac{105\pi^3+648\pi^2-2240\pi-2560}{(3\pi-8)^3} = 18.66866547\ldots$

For more details see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT9new.txt` .

We note that, by miracle, the (limiting, as $n$ goes to infinity) average, variance, and *any* higher moment, happened to be *Gosperable* so Maple is able to evaluate them in closed-form using the Maple command `sum`. We doubt whether this will happen for more rows, but we did not try.

We believe that the Maple package `SYT.txt` can be used to explore further and possibly suggest improvements to the already very impressive work in [CPP1] and [CPP2].

**Sample output**

The web-page of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/syt.html` ,

contains lots of output files. Let's just mention some highlights.

• If you want to see a computer-generated article with lots of explicit expressions (as rational functions of $n$) for the probability distribution of the occupant of cell $[1, i]$ in a (uniformly-at) random-generated Young tableau of rectangular shape with 2 rows and $n$ columns (i.e. of shape $[n, n]$) for all $i$ between 2 and 40, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT1.txt` .

If you want to see an abbreviated version, with only the *limiting distribution* as $n$ goes to infinity, but all the way to $i = 60$, as well as the expectation, variance, skewness, and kurtosis, see

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT1L.txt` .

If you want to see the plot of the occupancy distribution of the cell $[1, 40]$ as $n$ goes to $\infty$, look here:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYTpic2.html` .

• For the analogous output for 3-rowed rectangular shapes, (i.e. $(n, n, n)$), see, respectively https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT2.txt ,

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT2L.txt ,

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYTpic3.html .

For the analogous information for up to 8 rows (but with less data) see the above-mentioned front of this article.

• For testing the amazing Greene-Nijenhuis-Wilf algorithm vs. the exact results, see the output file

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYTsi1.txt .

To get lots of explicit expressions for sorting probabilities refer to the above web-page.

**References**


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https://sites.math.rutgers.edu/~zeilberg/mamarimY/Andre1983.pdf .
[Christian Krattenthaler noticed a very long time ago that on the second page (p. 326), last word on line 3: first → last] .

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