FORTY "STRANGE" COMPUTER-DISCOVERED [and COMPUTER-PROVED (of course!)] HYPERGEOMETRIC SERIES EVALUATIONS

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Abstract: We state and prove forty hypergeometric (${}_2F_1$) "strange" series evaluations that were found by a systematic search for 'nice' identities inspired by the simplified and annotated Zeilberger algorithm, implemented in the Maple package `twoFone`.

Foreword (by Human Doron Zeilberger)

There are infinitely many mathematical facts, some shallow but interesting (like $2 + 2 = 4$) and some deep (like the fact that the product of two random million-digit integers is some specific two-million-digit integer) but utterly boring (to humans, at least). A (human) *theorem* is an *interesting* (i.e. *simple*) fact that is not too trivial. Granted, a computer can discover, and even prove, lots of facts, but can it discover **theorems**?

Until now, WZ theory was able to *prove* 'nice' identities, but these had to be first *conjectured* (i.e. discovered) by humans. WZ theory could also *discover* lots of *new* 'theorems', usually much deeper than the human-discovered ones, by systematically running the Zeilberger algorithm on many different summands, finding, and simultaneously proving, recurrences satisfied by the definite sums. But the recurrences outputted are (usually) horrendous to human eyes. Humans like their facts to be *simple*. In [Z] I showed how to use the simplified Zeilberger algorithm [MZ] in order to look out for hypergeometric terms that yield recurrences of lower-than-expected order, in particular, of order one, which imply *closed-form evaluations*.

The novelty of the present compilation is not so much in its *substance*. It is very possible that all (or at least most) of these forty identities are either already known, or are derivable, via classical ${}_2F_1$ transformations, and more modern quadratic ones, from known evaluations. In particular Theorems 1 and 2 have recently been discovered and proved, by a completely different (and very interesting, analytical) approach by Robert Maier[M]. Others were conjectured by Bill Gosper and first proved by Ira Gessel and Dennis Stanton[GS].

Indeed the great novelty of this article is in its **form**. A computer discovering *ab initio*, humanly-nice results, worthy of the name *theorems*, **all by itself**, without any human intervention (except the initial one of programming the **ansatz** and the meta-algorithm). Let it serve as a **paradigm**

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and **iconic example** for future **computer-generated** *interesting* mathematics, as opposed to mere computer-assisted one.

Introduction

The classical *hypergeometric* series

$$F(a\,,\,b\,,\,c\,,\,x) \;=\; \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} x^k \quad ,$$

(where $(z)_k := z(z+1)(z+2)\cdots(z+k-1)$), that nowadays is more commonly denoted by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,;\, x\right) \quad ,$$

enjoys several *closed-form* evaluations. The most celebrated one is the *three-parameter* Gauss evaluation at $x = 1$ (see, e.g. [AAR])

$$F(a\,,\,b\,,\,c\,,\,1) \;=\; \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad . \qquad (Gauss)$$

Next comes Kummer's *two-parameter* exact evaluation at $x = -1$

$$F(a\,,\,b\,,\,1+a-b\,,\,-1) = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)} \quad . \qquad (Kummer)$$

and two other ones at $x = 1/2$, also due to Gauss. In addition, Gosper conjectured, and Gessel and Stanton[GS] proved, several 'strange' *one-parameter* evaluations at other values of x. Here I will state and prove *all* such **inequivalent** strange evaluations that are of the form

$$F(-an\,,\,bn+b_1\,,\,cn+c_1, x)$$

in the **range**

$$1 \le a \le 2 \quad , \quad -4 \le b \le 4 \quad , \quad -4 \le c \le 4 \quad ,$$

with a, b, c all **integers**. None of them is a specialization of the classical identities, and none of them can be derived from other ones in the list via the Euler and Pfaff transformations ([AAR] Theorem 2.2.5) and their iterations (up to depth 3).

They were found by running the Maple package `twoFone` written by my beloved mentor, Doron Zeilberger, that accompanies his article [Z]. I ran out of time and memory for such identities when b and/or c were allowed to be 5. It would be interesting to know whether there exist infinitely many such strange evaluations. Another curious surprise was that the numbers b_1, c_1, that are, in general, along with x, solutions of a system of *polynomial* equations, and hence are only guaranteed to be *algebraic*, turn out, at least in all the examples discovered so far, to be *rational numbers.* On the other hand, x is sometimes irrational, but so far, it was, at worst, a *quadratic* irrationality.

The Forty Identities

The proofs below are given in the famous, succinct, WZ (Wilf-Zeilberger) format([WZ], see also [PWZ]), that only states a rational function, the so-called *certificate*, of (n,k). Recall that in order to translate it into a traditional proof, one has to divide the summand by the right side, and call it $F(n,k)$. Calling the supplied certificate $R(n,k)$, one defines $G(n,k) := R(n,k)F(n,k)$ and verifies the purely-routine high-school algebra identity

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k) \quad . \qquad (WZ)$$

(WZ) is indeed routine, in every given case, since dividing throughout by $F(n,k)$ and simplifying, yields an identity amongst *rational functions.* Finally summing (WZ) over k proves that $\sum_k F(n,k)$ is identically constant, as a function of n, and the proof is completed by checking that it equals 1 when $n = 0$.

Theorem 1 :

$$F(-n\,,\,-4n-1/2\,,\,-3n\,,\,-1) \;=\; \left(\frac{2^6}{3^3}\right)^n \cdot \frac{(3/8)_n(5/8)_n}{(1/3)_n(2/3)_n} \quad .$$

Proof :

$$\frac{-2(296n^3 - 216n^2k + 771n^2 + 651n - 372nk + 52nk^2 - 155k + 177 - 4k^3 + 45k^2)(3n-k+1)k}{(8n+9-2k)(8n+7-2k)(8n+5-2k)(8n+3-2k)(n-k+1)} \quad \square$$

Theorem 2 :

$$F(-n\,,\,-4n-5/2\,,\,-3n-1\,,\,-1) \;=\; \left(\frac{2^6}{3^3}\right)^n \cdot \frac{(7/8)_n(9/8)_n}{(2/3)_n(4/3)_n} \quad .$$

Proof :

$$(296n^3 + 1215n^2 - 216n^2k + 1660n + 52nk^2 - 586nk + 756 - 394k - 4k^3 + 71k^2)\cdot$$

$$\frac{-2(3n-k+2)k}{(8n+13-2k)(8n+11-2k)(8n+9-2k)(8n+7-2k)(n-k+1)} \quad \square$$

Theorem 3 :

$$F(-n\,,\,-3n-1/2\,,\,-4n\,,\,4) \;=\; \left(\frac{-3^3}{2^4}\right)^n \cdot \frac{(1/3)_n(5/6)_n}{(1/4)_n(3/4)_n} \quad .$$

Proof :

$$\frac{(344n^3 - 216n^2k + 768n^2 + 522n - 282nk + 34nk^2 - 75k + 104 + 13k^2)(4n-k+1)k}{4(6n+7-2k)(6n+5-2k)(6n+3-2k)(n-k+1)(3n+1)} \quad \square.$$

Theorem 4 :

$$F(-n\,,\,-3n-3/4\,,\,-4n-1\,,\,-8)\;=\;\left(\frac{3^3}{2^2}\right)^n\cdot\frac{(2/3)_n(7/12)_n}{(1/2)_n(3/4)_n}.$$

Proof :

$$\frac{-(2208n^3-1488n^2k+6272n^2+5762n-2620nk+252nk^2-1082k+1707+167k^2)(4n-k+2)k}{12(12n+15-4k)(12n+11-4k)(12n+7-4k)(n-k+1/(3n+2)}\quad\square$$

Theorem 5 :

$$F(-n\,,\,-3n-3/2\,,\,-4n-2\,,\,-8)\;=\;\left(\frac{3^3}{2^2}\right)^n\cdot\frac{(5/6)_n(7/6)_n}{(3/4)_n(5/4)_n}\quad.$$

Proof :

$$-(1/24)\frac{(184n^2+492n-124nk+326+21k^2-167k)(4n-k+3)k}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n-k+1)}\quad\square$$

Theorem 6 :

$$F(-n\,,\,-3n-1/4\,,\,-4n\,,\,-8)\;=\;\left(\frac{3^3}{2^2}\right)^n\cdot\frac{(1/3)_n(5/12)_n}{(1/2)_n(1/4)_n}.$$

Proof :

$$\frac{(-1/4)(736n^3-496n^2k+1600n^2+1058n+84nk^2-624nk-149k+200+27k^2)(4n-k+1)k}{(12n+13-4k)(12n+9-4k)(12n+5-4k)(n-k+1)(3n+1)}\quad\square$$

Theorem 7 :

$$F(-n\,,\,-3n-1/2\,,\,-4n\,,\,-8)\;=\;\left(\frac{3^3}{2^2}\right)^n\cdot\frac{(1/3)_n(2/3)_n}{(1/4)_n(3/4)_n}.$$

Proof :

$$(-1/32)(2208n^4-1488n^3k+6640n^3+252n^2k^2-2988n^2k+7104n^2+248nk^2-1812nk+3156n+53k^2-321k+478)\cdot$$

$$\frac{(4n-k+1)k}{(6n+7-2k)(6n+5-2k)(6n+3-2k)(n-k+1)(3n+2)(3n+1)}\quad\square$$

Theorem 8 :

$$F(-n\,,\,-3n-5/4\,,\,-4n-1\,,\,-8)\;=\;\left(\frac{3^3}{2^2}\right)^n\cdot\frac{(2/3)_n(13/12)_n}{(1/2)_n(5/4)_n}.$$

Proof :

$$\frac{(-1/4)(736n^3-496n^2k+2336n^2+2382n-956nk+84nk^2-414k+767+55k^2)(4n-k+2)k}{(12n+17-4k)(12n+13-4k)(12n+9-4k)(n-k+1)(3n+2)}\quad\square$$

Theorem 9 :

$$F(-n\,,\,-3n-3/2\,,\,-4n-2\,,\,4)\;=\;\left(\frac{-3^3}{2^4}\right)^n\cdot\frac{(4/3)_n(5/6)_n}{(3/4)_n(5/4)_n}.$$

Proof :

$$\frac{(1/12)(1032n^3-648n^2k+4024n^2+5202n-1690nk+102nk^2-1103k+2228+135k^2)(4n-k+3)k}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n-k+1)(3n+4)}\quad\square$$

Theorem 10 :

$$F(-n\,,\,-3n-7/4\,,\,-4n-2\,,\,-8)\;=\;\left(\frac{3^3}{2^2}\right)^n\frac{(4/3)_n(11/12)_n}{(3/2)_n(3/4)_n}.$$

Proof :

$$\frac{(-1/12)(2208n^3+9216n^2-1488n^2k+12806n+252nk^2-4112nk+5924-2839k+335k^2)(4n-k+3)k}{(12n+19-4k)(12n+15-4k)(12n+11-4k)(n-k+1)(3n+4)}\quad\square$$

Theorem 11 : Let ω be a primitive cubic root of unity (i.e. $\omega\;=\;(1\pm\sqrt{3}i)/2$).

$$F(-n\,,\,-3n-1\,,\,-2n\,,\,\omega)\;=\;(-(3/2)\omega+3/4)^n\cdot\frac{(2/3)_n}{(1/2)_n}\quad.$$

Proof :

$$(129n^2+289n-2\omega n+160-2\omega-88kn+14k\omega n-101k+19k\omega+13k^2-5\omega k^2)\cdot$$

$$\frac{(-1/129)(-5+13\omega)(2n-k+1)k}{(3n+4-k)(3n+3-k)(3n+2-k)(n-k+1)}\quad\square$$

Theorem 12 :

$$F(-n\,,\,-3n-1/2\,,\,-n+1/2\,,\,4)\;=\;(-3^3)^n.$$

Proof :

$$\frac{(1/9)k(2n+1-2k)(84n^2+172n-60nk+12k^2-64k+87)}{(6n+7-2k)(6n+5-2k)(6n+3-2k)(n+1-k)}\quad\square$$

Theorem 13 :

$$F(-n\,,\,-3n-3/2\,,\,-n+1/2\,,\,4)\;=\;(-3^3)^n\cdot\frac{(5/6)_n(7/6)_n}{(1/2)_n(3/2)_n}\quad.$$

Proof :

$$\frac{(1/3)(28n^2+76n-20nk+51-28k+4k^2)(2n+1-2k)k}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n+1-k)}\quad\square$$

Theorem 14 : Let $\alpha = 9 \pm 4\sqrt{5}$**.**

$$F(-n\,,\,-3n-1/2\,,\,2n+3/2\,,\,\alpha) \;=\; \left(\frac{2^7}{5^3}(1+\alpha)\right)^n \cdot \frac{(3/4)_n(5/4)_n}{(4/5)_n(6/5)_n} \quad .$$

Proof :

$$(79794+388827n+706989n^2+566136n^3+167796n^4-4518k-4875nk-51972k^2+11429\alpha nk+$$
$$3754\alpha k+2132\alpha k^2+496k^4\alpha+26472k^3-3696k^4-6816kn^3-6174kn^2-127434k^2n-72852k^2n^2+30768k^3n$$
$$-3336n^3\alpha-9466\alpha n^2-8839\alpha n-2718\alpha+10998k\alpha n^2+1060k^2\alpha n^2-2192k^3\alpha n+3336kn^3\alpha+3058k^2\alpha n-2584k^3\alpha)\cdot$$
$$\frac{-(1/74576)(31\alpha-327)k}{(6n+7-2k)(6n+5-2k)(6n+3-2k)(n-k+1)(4n+3+2k)(\alpha-1)}. \quad \square$$

Theorem 15 : Let $\alpha = 9 \pm 4\sqrt{5}$**.**

$$F(-n\,,\,-3n-3/2\,,\,2n+5/2\,,\,\alpha) \;=\; \left(\frac{2^7}{5^3}(1+\alpha)\right)^n \cdot \frac{(5/4)_n(7/4)_n}{(7/5)_n(8/5)_n} \quad .$$

Proof :

$$(384115+1282296n+1575413n^2+845796n^3+167796n^4-86193k-153105nk-77610k^2+26963\alpha nk+$$
$$13383\alpha k+730\alpha k^2+496k^4\alpha+34264k^3-3696k^4-6816kn^3-$$
$$74602kn^2-154458k^2n-72852k^2n^2+30768k^3n-3336n^3\alpha-12126\alpha n^2-14517\alpha n-$$
$$5745\alpha+17114k\alpha n^2+1060k^2\alpha n^2-2192k^3\alpha n+3336kn^3\alpha+1842k^2\alpha n-2984k^3\alpha)\cdot$$
$$\frac{k(-1/74576)(31\alpha-327)}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n-k+1)(4n+5+2k)(\alpha-1)} \quad \square$$

Theorem 16 : Let $\alpha = 2 \pm \sqrt{3}$**.**

$$F(-n\,,\,-2n-1/2\,,\,-4n-1\,,\,4\alpha) \;=\; (3/4-(3/2)\alpha)^n \cdot \frac{(2/3)_n}{(1/2)_n} \quad .$$

Proof :

$$(488n^2+978n-20\alpha n+472-23\alpha+94k^2-25k^2\alpha-420kn+52kn\alpha-422k+72k\alpha)\cdot$$
$$\frac{(1/732)(-68+19\alpha)(4n+2-k)k}{(4n+5-2k)(4n+3-2k)(n-k+1)(3n+2)} \quad \square$$

Theorem 17 :

$$F(-n\,,\,-2n-1/3\,,\,-3n\,,\,9) \;=\; (-2^2)^n \cdot \frac{(1/2)_n}{(1/3)_n} \quad .$$

Proof :
$$\frac{(1/9)k(3n-k+1)(45n^2-21nk+72n-11k+25)}{(6n+7-3k)(6n+4-3k)(n-k+1)(2n+1)} \quad \square.$$

Theorem 18 :
$$F(-n\,,\,-2n-2/3\,,\,-3n-1\,,\,9)\;=\;(-2^2)^n\cdot\frac{(5/6)_n}{(2/3)_n} \quad .$$

Proof :
$$\frac{(1/6)(15n+19-7k)(3n-k+2)k}{(6n+8-3k)(6n+5-3k)(n-k+1)} \quad \square$$

Theorem 19 :
$$F(-n\,,\,-n-1/2\,,\,4n+9/2\,,\,1/5)\;=\;\left(\frac{2^{14}}{5^6}\right)^n\cdot\frac{(9/8)_n(11/8)_n(13/8)_n(15/8)_n}{(6/5)_n(9/5)_n(13/10)_n(17/10)_n} \quad .$$

Proof :
$$(3036n^4+1256n^3k+16960n^3+35275n^2+4780n^2k-1184n^2k^2+6180nk-$$
$$3440nk^2-544nk^3+32445n+2720k+11154-2480k^2-64k^4-800k^3)\cdot$$
$$\frac{(-5/16)k}{(2n+3-2k)(n-k+1)(8n+13+2k)(8n+11+2k)(8n+9+2k)} \quad \square$$

Theorem 20 :
$$F(-n\,,\,-n+1/2\,,\,4n+3/2\,,\,1/5)\;=\;\left(\frac{2^{14}}{5^6}\right)^n\cdot\frac{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}{(3/5)_n(2/5)_n(9/10)_n(11/10)_n} \quad .$$

Proof :
$$(3036n^4+1256n^3k+7852n^3+7409n^2+2844n^2k-1184n^2k^2+2252nk-1344nk^2+2993n-$$
$$544nk^3-336k^2+592k+430-352k^3-64k^4)\cdot$$
$$\frac{(-5/16)k}{(2n+1-2k)(n-k+1)(8n+7+2k)(8n+5+2k)(8n+3+2k)} \quad \square$$

Theorem 21 : Let $\beta\;=\;(3\pm\sqrt{5})/2$.
$$F(-n\,,\,2n+1\,,\,-3n\,,\,\beta)\;=\;\left(\frac{5^2}{3^3}(1+\beta)\right)^n\cdot\frac{(2/5)_n(3/5)_n}{(1/3)_n(2/3)_n} \quad .$$

Proof :
$$(1376n+308+2026n^2+958n^3-60k-23k^3\beta+38k^2\beta-91k\beta+139\beta n+87\beta n^2$$

$$-201k\beta n-87k\beta n^2+38k^2\beta n+48k^3+4k^2-122kn-110kn^2+4k^2n+52\beta)\cdot$$

$$\frac{(1/4790)k(3n-k+1)(-78+17\beta)}{(n-k+1)(5n+3)(5n+2)(n+1)(2n+1)} \quad \square$$

Theorem 22 : Let $\beta = (3\pm\sqrt{5})/2$**.**

$$F(-n\,,\,2n+2\,,\,-3n-1\,,\,\beta) = \left(\frac{5^2}{3^3}(1+\beta)\right)^n \cdot \frac{(4/5)_n(6/5)_n}{(2/3)_n(4/3)_n} \quad .$$

Proof :

$$(4179n+3463n^2+958n^3+24k-317k\beta n-87k\beta n^2+48k^3+54k^2-109kn-110kn^2+4k^2n+96\beta+$$

$$1674+183\beta n+87\beta n^2+38k^2\beta n-23k^3\beta+34k^2\beta-251k\beta)\cdot$$

$$\frac{(1/4790)k(3n-k+2)(-78+17\beta)}{(n-k+1)(5n+6)(5n+4)(2n+3)(n+1)} \quad \square$$

Theorem 23 : Let ϕ **be the Golden ratio** $(1+\sqrt{5})/2$**.**

$$F(-n\,,\,2n+1\,,\,4n+2\,,\,\phi^{-1}) = \left(\frac{2^6}{5^3}(2-\phi^{-1})\right)^n \cdot \frac{(3/4)_n(5/4)_n}{(4/5)_n(6/5)_n} \quad .$$

Proof :

$$(2815n+3661n^2+1558n^3+14k^3\phi^{-1}-148\phi^{-1}n-84\phi^{-1}n^2+579k+183k\phi^{-1}n+$$

$$1567kn+1013kn^2+88k\phi^{-1}+37k^2\phi^{-1}n+34k^2\phi^{-1}+$$

$$270k^2n+84k\phi^{-1}n^2+712+206k^2+39k^3-64\phi^{-1})\cdot$$

$$\frac{(-1/779)k(14\phi^{-1}-25)}{(n-k+1)(4n+4+k)(4n+3+k)(4n+2+k)(-1+\phi^{-1})} \quad \square$$

Theorem 24 : Let ϕ **be the Golden ratio** $(1+\sqrt{5})/2$**.**

$$F(-n\,,\,2n+2\,,\,4n+4\,,\,\phi^{-1}) = \left(\frac{2^6}{5^3}(2-\phi^{-1})\right)^n \cdot \frac{(5/4)_n(7/4)_n}{(7/5)_n(8/5)_n} \quad .$$

Proof :

$$(7628n+5998n^2+1558n^3+14k^3\phi^{-1}-178\phi^{-1}n$$

$$-84\phi^{-1}n^2+3188+1762k+309k\phi^{-1}n+2697kn+1013kn^2+253k\phi^{-1}+$$

$$37k^2\phi^{-1}n+79k^2\phi^{-1}+270k^2n+84k\phi^{-1}n^2+387k^2+39k^3-94\phi^{-1})\cdot$$

$$\frac{(-1/779)k(14\phi^{-1}-25)}{(n-k+1)(4n+6+k)(4n+5+k)(4n+4+k)(-1+\phi^{-1})} \quad \square$$

Theorem 25 : Let $\gamma \ = \ 3 \pm 2\sqrt{2}$.

$$F(-2n\,,\,-4n-1/2\,,\,-2n+1/2\,,\,\gamma) \ = \ (-16+96\gamma)^n \frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n} \quad .$$

Proof :

$$(25434+137162n+271600n^2+233440n^3+73472n^4+5216k^2\gamma n^2+10144k^2\gamma n-21148k\gamma n-$$

$$6208k\gamma n^3-20304k\gamma n^2-1264k^3\gamma n-142224kn^2-58944kn^3-111324kn+14304k^2n^2+$$

$$22536k^2n-1776k^3n-1372k^3+8594k^2+88k^4+4834k^2\gamma-7067k\gamma-1204k^3\gamma-28145k+3837\gamma+$$

$$104k^4\gamma+13852\gamma n+6208\gamma n^3+16240\gamma n^2)\cdot$$

$$\frac{(-1/1148)(-89+13\gamma)(4n+1-2k)k}{(8n+9-2k)(8n+7-2k)(8n+5-2k)(8n+3-2k)(2n-k+1)(2n-k+2)(\gamma-1)} \quad \square$$

Theorem 26 : Let $\delta \ = \ 5 \pm 3\sqrt{3}$.

$$F(-2n\,,\,-4n-1/3\,,\,2/3\,,\,2\delta) \ = \ (3^3(26\delta+5))^n \ \cdot \ \frac{(7/12)_n}{(3/4)_n} \quad .$$

Proof :

$$(2059512+12099448n+766292\delta n+25671232n^2+23410464n^3+7767936n^4+933000\delta n^2-$$

$$1239770k\delta n-404352kn^3\delta+506562k^2\delta n+268020k^2\delta n^2-62100k^3\delta n-1257180k\delta n^2$$

$$-2121648k-5589216kn^3-12542136kn^2-9090980kn+1684440k^2n^2+2508684k^2n$$

$$-240624k^3n-56934k^3\delta+232953k^2\delta-390660k\delta+370656n^3\delta$$

$$+205020\delta+13446k^4-178812k^3+904542k^2+4941k^4\delta)\cdot$$

$$\frac{(1/364122)k(-1+3k)(61\delta-776)}{(12n+13-3k)(12n+10-3k)(12n+7-3k)(12n+4-3k)(2n-k+1)(2n-k+2)2\delta-1)} \quad \square$$

Theorem 27 : Let $\delta \ = \ 5 \pm 3\sqrt{3}$.

$$F(-2n\,,\,-4n-2/3\,,\,4/3\,,\,2\delta) = (3^3(26\delta+5))^n \ \cdot \ \frac{(5/12)_n}{(5/4)_n} \quad .$$

Proof :

$$(3710744+18504312n+607916\delta n+33746032n^2+26726016n^3+7767936n^4+661656\delta n^2-$$

$$1315958k\delta n-404352kn^3\delta+533802k^2\delta n+268020k^2\delta n^2-62100k^3\delta n-1280676k\delta n^2-$$

$$3169096k-5589216kn^3-14213784kn^2-11770892kn+1684440k^2n^2+2829540k^2n-240624k^3n$$

$$-60696k^3\delta+260835k^2\delta-439228k\delta+235872n^3\delta+182668\delta$$

$$+13446k^4 - 200988k^3 + 1159494k^2 + 4941k^4\delta)\cdot$$

$$\frac{(1/364122)k(1+3k)(61\delta - 776)}{(12n+14-3k)(12n+11-3k)(12n+8-3k)(12n+5-3k)(2n-k+1)(2n-k+2)(2\delta-1)} \quad \square$$

Theorem 28 : Let $\alpha \ = \ 2 \pm \sqrt{3}$**.**

$$F(-2n\,,\, -3n-1/4\,,\, -4n\,,\, 4\alpha) \ = \ ((39/2)\alpha - 21/4)^n \cdot \frac{(5/12)_n}{(1/4)_n} \quad .$$

Proof :

$$(171978n + 238000n^2 + 104736n^3 - 12496k\alpha n^2 - 32440k\alpha n + 7780k^2\alpha n + 17488\alpha n^2 +$$

$$30900\alpha n + 8407k^2\alpha - 1216k^3\alpha - 18581k\alpha + 39182 + 13262\alpha$$

$$-81808kn^2 + 15490k^2 - 1792k^3 - 43403k - 122224kn + 21112k^2n)\cdot$$

$$\frac{(-1/3273)k(4n-k+1)(397\alpha - 1484)}{(12n+13-4k)(12n+9-4k)(12n+5-4k)(2n-k+1)(2n-k+2)} \quad \square$$

Theorem 29 :

$$F(-2n\,,\, -3n+1\,,\, -2n+2\,,\, -2) \ = \ (3^3)^n \cdot (-2n+1) \quad .$$

Proof :

$$\frac{(-1/9)k(2n-k-1)(39n^2+55n-21kn-14k+3k^2+18)}{(3n+2-k)(3n+1-k)(3n-k)(2n+2-k)} \quad \square$$

Theorem 30 :

$$F(-2n\,,\, -3n\,,\, -2n+1/2\,,\, 1/4) \ = \ \left(\frac{3^3}{2^6}\right)^n \cdot \frac{(1/3)_n(2/3)_n}{(1/4)_n(3/4)_n} \quad .$$

Proof :

$$\frac{(-2/3)(74n^3+168n^2-57kn^2-87kn+124n+12k^2n-32k+30-k^3+9k^2)(4n+1-2k)k}{(3n+3-k)(3n+2-k)(3n+1-k)(2n+1-k)(2n+2-k)} \quad \square$$

Theorem 31 : Let $\eta \ = \ -7 \pm 4\sqrt{3}$**.**

$$F(-2n\,,\, -3n-1/4\,,\, -n+3/4\,,\, \eta) \ = \ (312\eta + 24)^n \cdot \frac{(5/12)_n}{(1/4)_n} \quad .$$

Proof :

$$(485694n + 611008n^2 + 248736n^3 - 5600k^2\eta n + 11504k\eta n^2 + 23720\eta nk - 11880\eta n-$$

$$6512\eta n^2+752k^3\eta+22744k^2+11851\eta k-5432\eta k^2-204544nk-79751k-2416k^3$$

$$+28576k^2n-126928kn^2-5299\eta+124358)\cdot$$

$$\frac{(1/124368)(47\eta+809)(4n+1-4k)k}{(12n+13-4k)(12n+9-4k)(12n+5-4k)(2n-k+1)(2n-k+2)(\eta-1)} \quad \square$$

Theorem 32 :

$$F(-2n\,,\,-2n+1/3\,,\,-n+5/6\,,\,-1/8\,)\;=\;\left(\frac{3^3}{2^4}\right)^n\cdot\frac{(1/3)_n}{(1/6)_n}\quad .$$

Proof :

$$\frac{(-4/27)(132n^2-60nk+202n+74+9k^2-45k)(6n+1-6k)k}{(6n+5-3k)(6n+2-3k)(2n-k+1)(2n-k+2)} \quad \square$$

Theorem 33 : Let $\zeta\;=\;-1\pm\sqrt{2}$.

$$F(-2n\,,\,1/2\,,\,-4n\,,\,2\zeta)\;=\;\frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n}\quad .$$

Proof :

$$\frac{(-1/2)(3+\zeta)(6\zeta n-2n+4\zeta-3k^2-k^2\zeta+8kn+6k)(4n-k+1)k}{(2n-k+2)(2n-k+1)(8n+5)(8n+3)} \quad \square$$

Theorem 34 : Let ϕ be the Golden ratio $(1+\sqrt{5})/2$ then

$$F(-2n\,,\,2n+1\,,\,-3n+1/2\,,\,\phi)\;=\;\left(\frac{5^2}{3^3}(4\phi+3)\right)^n\cdot\frac{(3/10)_n(7/10)_n}{(1/6)_n(5/6)_n}\quad .$$

Proof :

$$(1678n+2708n^2+1448n^3+136k^2\phi n-264k\phi n^2-364k\phi n-118k\phi+324\phi n$$

$$+264\phi n^2-26k^3\phi+96k^2\phi+343+96\phi-267k-19k^2+24k^3-42k^2n-536kn^2-750kn)\cdot$$

$$\frac{(2/905)k(6n+1-2k)(-43+16\phi)}{(2n+1)(2n-k+2)(2n-k+1)(10n+7)(10n+3)} \quad \square$$

Theorem 35 : Let ϕ be the Golden ratio $(1+\sqrt{5})/2$.

$$F(-2n\,,\,2n+1\,,\,-3n\,,\,\phi)\;=\;\left(\frac{5^2}{3^3}(4\phi+3)\right)^n\cdot\frac{(2/5)_n(3/5)_n}{(1/3)_n(2/3)_n}.$$

Proof :

$$(1334n+1716n^2+724n^3-270k\phi n+68k^2\phi n-132k\phi n^2-125k\phi+214\phi n+132\phi n^2$$

$$-13k^3\phi+68k^2\phi+82\phi-177k-21k^2+12k^3-21k^2n-268kn^2-433kn+342)\cdot$$

$$\frac{(1/905)k(3n-k+1)(-43+16\phi)}{(2n-k+2)(2n-k+1)(5n+3)(5n+2)(n+1)} \quad \square$$

Theorem 36 : Let $\zeta_1 \;=\; 1 \pm \sqrt{2}$**.**

$$F(-2n\,,\,2n+1\,,\,-2n+1/2\,,\,(1/2)\zeta_1) \;=\; (4+8\zeta_1)^n \cdot \frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n} \quad .$$

Proof :

$$(56n^2+72n+8\zeta_1 n+24+5\zeta_1-6k^2-16kn-12k+4k^2\zeta_1-8\zeta_1 kn-6\zeta_1 k)\cdot$$

$$\frac{(2/7)k(4n+1-2k)(-4+\zeta_1)}{(2n-k+2)(2n-k+1)(8n+5)(8n+3)} \square$$

Theorem 37 : Let $\alpha \;=\; 2 \pm \sqrt{3}$**.**

$$F(-2n\,,\,2n+1\,,\,-n+3/4\,,\,(1/4)\alpha) = (3-6\alpha)^n \cdot \frac{(5/12)_n}{(1/4)_n} \quad .$$

Proof :

$$\frac{(1/3)(8n+5+\alpha-2k\alpha+2k)(-5+\alpha)(4n+1-4k)k}{(2n-k+2)(2n-k+1)(12n+5)} \quad \square$$

Theorem 38 :

$$F(-2n\,,\,3n+1\,,\,-2n+2\,,\,2/5) \;=\; \left(\frac{5^3}{3^3}\right)^n \cdot (-2n+1) \quad .$$

Proof :

$$\frac{(-1/25)(147n^2+33nk+179n+3k^2+23k+50)(2n-k-1)k}{n(2n+2-k)(3n+2)(3n+1)} \quad \square$$

Theorem 39 : Let $\zeta_2 \;=\; 1 \pm \sqrt{3}/3$**.**

$$F(-2n\,,\,4n+2\,,\,4/3\,,\,(2/3)\zeta_2) \;=\; (-9+18\zeta_2)^n \cdot \frac{(5/12)_n}{(5/4)_n} \quad .$$

Proof :

$$(352n^2+432n+168\zeta_2 n+116+138\zeta_2-2k^2+9\zeta_2 k^2-$$

$$288\zeta_2 nk+152nk+126k-237\zeta_2 k)\cdot$$

$$\frac{(-1/88)(-16+9\zeta_2)(1+3k)k}{(2n-k+2)(2n-k+1)(12n+5)(4n+3)} \quad \square$$

Theorem 40 : Let $\zeta_2 = 1 \pm \sqrt{3}/3$.

$$F(-2n\,,\,4n+1\,,\,2/3\,,\,(2/3)\zeta_2) \;=\; (-9+18\zeta_2)^n \cdot \frac{(7/12)_n}{(3/4)_n} \quad .$$

Proof :

$$(352n^2+264n+264\zeta_2 n+20+174\zeta_2-2k^2+9\zeta_2 k^2-288\zeta_2 nk+152nk+102k-195\zeta_2 k)\cdot$$

$$\frac{(-1/88)(-16+9\zeta_2)(-1+3k)k}{(2n-k+2)(2n-k+1)(12n+7)(4n+1)} \quad \square$$

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