USING THE JACOBI-TRUDI FORMULA TO COMPUTE STIRLING DETERMINANTS

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PART I: Theory

The *unsigned Stirling numbers of the first kind* $\left[{n \atop k}\right]$ enumerate permutations of n elements with k disjoint cycles. They also arise as coefficients of the rising factorial, i.e.,

$$x(x+1)(x+2)\cdots(x+n-1)=\sum_{k=0}^{n}\left[{n \atop k}\right]x^k.$$

The *Stirling numbers of the second kind* $\left\{{n \atop k}\right\}$ enumerate the number of ways to partition a set of n objects into k non-empty subsets. They also arise as coefficients of the falling factorial, i.e.,

$$\frac{x^k}{x(x-1)(x-2)\cdots(x-k)}=\sum_{n=1}^{\infty}\left\{{n \atop k}\right\}x^n.$$

Assume a and b to be non-negative integers. Our specific interest lies in computing the determinants of the following $n\times n$ matrices

$$M_n(a,b)=\left(\left[{i+a \atop j+b}\right]\right)_{1\le i,j\le n} \qquad \text{and} \qquad N_n(a,b)=\left(\left\{{i+a \atop j+b}\right\}\right)_{1\le i,j\le n}.$$

Denote $\beta_n(a,b)=\det(M_n(a,b))$ and $\gamma_n(a,b)=\det(N_n(a,b))$. Let $[a]=\{1,2,\dots,a\}$ be the integer interval. Given a partition λ, the *Schur functions* can be given by

$$s_\lambda(\xi_1,\dots,\xi_a)=\frac{\det\left(\xi_i^{\lambda_j+a-j}\right)_{1\le i,j\le a}}{\det\left(\xi_i^{a-j}\right)_{1\le i,j\le a}}.$$

We are now ready to state our results.

Theorem 1. *For $a,b,n\in\mathbb{Z}_{\ge 0}$ and $b\le a$, the sequence $\beta_n(a,b)$ has a rational generating function, in the variable q, with linearly factored denominator having the form*

$$\prod_{\substack{i_1<i_2<\cdots<i_b\\ i_1,i_2,\dots,i_b\in[a]}}\left(1-\frac{a!}{i_1i_2\cdots i_b}q\right).$$

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Theorem 2. *Denote $(n^c) = (n, n, \dots, n) \vdash cn$ to be a partition of cn. Then, we have*

$$\beta_n(a,b) = s_{(n^{a-b})}(1,2,\dots,a) \qquad \text{and} \qquad \gamma_n(a,b) = s_{((a-b)^n}(1,2,\dots,a)).$$

Proof. First, observe that the Stirling numbers of the first and second kinds are the respective specializations of e_m and h_m. Both assertions follow from the Jacobi-Trudi and Nägelsbach-Kostka identities

$$s_\lambda(\boldsymbol{\xi}) = \det(e_{\lambda'_j+j-i}(\boldsymbol{\xi}))_{1\le i,j\le n} = \det(h_{\lambda_j+j-i}(\boldsymbol{\xi}))_{1\le i,j\le n},$$

where λ' is the *conjugate* partition to λ and $e_m(\boldsymbol{\xi}), h_m(\boldsymbol{\xi})$ are the elementary and the complete homogeneous symmetric functions, respectively. □

Theorem 3. *Let $A_i = (-1)^{a-i} \cdot \frac{i^a}{a!}\binom{a}{i}$, for $i \in [a]$. Then, we have the "explicit" expressions*

$$\beta_n(a+b,b) = (-1)^{\binom{a}{2}} \sum_{1\le i_1<\cdots<i_a\le a+b} \prod_{\ell=1}^{a} A_{i_\ell} \cdot \prod_{\substack{\ell_u<\ell_v \\ \ell_u,\ell_v\in\{i_1,\dots,i_a\}}} (i_{\ell_u} - i_{\ell_v})^2 \cdot \prod_{\ell=1}^{a} i_\ell^{n-a+1} = \gamma_a(n+b,b).$$

Remark. R. Stanley informed the first author that our identity for special case $\beta_n(a+1,a)$ looks like its should be equivalent to Exercise 7.4 in [2] (see also references therein).

PART II: Computations

Using the Jacobi-Trudi formula mentioned in Theorem 2 in Part I (Eq. (I.3.5) in [1], page 41) we computed explicit expressions for $\beta_n(a,b)$, and also the explicit generating functions $\sum_{n=0}^{\infty} \beta_n(a,b)q^n$ (that are always rational functions of q), for **all** $10 \ge n \ge a \ge b \ge 0$.

The output file is

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oStirlingDet1.txt`

It was generated by executing the command

`Paper1(10,n,q):`

in the Maple package accompanying this article, that can be gotten from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/StirlingDet.txt`

References

[1] Ian G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Clarendon Press, Oxford, 1995..
[2] R. P. Stanley, *Enumerative combinatorics, Vol. 2,* Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, Cambridge, 1999. xii+581 pp.

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