Explicit Expressions for the Variance and Higher Moments of the Size of a Simultaneous Core Partition and its Limiting Distribution

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Dedicated to William Y.C. "Bill" Chen, the tireless apostle of enumerative and algebraic combinatorics in China (and beyond)

VERY IMPORTANT

As in all our joint papers, the main point is not the article, but the accompanying Maple package, `stCore`, that may be downloaded, free of charge, from the webpage of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/stcore.html` ,

where the readers can also find sample input and output files, that they are welcome to extend using their own computers.

Introduction

Many (perhaps most) combinatorial *statistics* (e.g. the number of Heads in tossing a coin n times, the number of inversions of an n-permutation [and, more generally, the number of occurrences of any pattern in an n-permutation]), are *asymptotically normal*, which means that if you denote it by X_n, figure out the *average*, $a_n := \mathbf{E}[X_n]$, and then figure out the *variance*, let's call it $m_2(n) := \mathbf{E}[(X_n - a_n)^2]$, then the *centralized and standardized* version $Z_n := (X_n - a_n)/\sqrt{m_2(n)}$ tends, as $n \to \infty$, to the good old *normal distribution* (aka *Gaussian distribution*) whose probability density function is $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Our favorite way ([Z1][Z2]) of proving this is *automatically*, by using *symbol-crunching* to compute (at least the leading terms) of the general moment $m_r(n)$ and then prove, (*automatically*, of course) that $\lim_{n\to\infty} \frac{m_r(n)}{m_2(n)^{r/2}}$, equals 0 for r odd, and $r!/((r/2)!\,2^{r/2})$, for r even, the famous moments of the normal distribution.

But there are numerous exceptions! The most notable is the statistics "length of the longest increasing subsequence" defined over the set of n-permutations, where the intriguing Tracy-Widom distribution shows up. Another example is the subject matter of the present article, the random variable "size", defined over the set of (s,t)-core partitions (see below for definition), where s and t are relatively prime positive integers. One of us (DZ) pledges a 100 dollars donation to the OEIS Foundation, in honor of the first prover(s), for meeting the following challenge.

First Challenge: Prove, rigorously, that the scaled limiting distribution (see below) (as (s,t) both go to infinity, and $s-t$ is a fixed constant) of the combinatorial random variable "size" defined on the set of (s,t)-core partitions is given by the continuous random variable

$$\sum_{k=1}^{\infty} \frac{z_k^2 + \tilde{z}_k^2}{4\pi^2 k^2} \quad ,$$

where z_k and $\tilde{z}_k$ are jointly independent sequences of independent standard normal random variables.

One of us (SBE) verified that the first nine (standardized) moments of "size of an (s,t)-core partition", (as $(s,t) \to \infty$) converge to the corresponding moments of the above continuous distribution, and it is virtually certain that this is true in general (see below for details).

This distribution is mentioned in [DGP], eq. (2.4), where it is called $U_{VS}(1)$.

But the main purpose of this article is to show the power of *symbol-crunching* in deriving new mathematical knowledge. We will state deep new (polynomial) expressions for the first six moments of the random variable *size of an (s,t)-core partition*, and the first nine moments for the special case of $(s, s+1)$-core partitions. From the *"religious-fanatical"* viewpoint of the current "mainstream" mathematician, they are "just" conjectures, but nevertheless, they are **absolutely certain** (well, at least as absolutely certain as most proved theorems). We also briefly indicate how we derived these expressions, and indicate, for those *obtuse* mathematicians who would like to see mathematical proofs, how they may possibly be proved.

We are not offering any prizes for just any old proofs, but we will be delighted to donate another \$100 to the OEIS Foundation for meeting the following challenge.

Second Challenge: Come up with a "hand-waving" and "soft" (yet rigorous!) *a priori* reason (whose length is not to exceed two pages) why the average, and any finite moment, must be a polynomial in s and t, and also come up with a "soft" (but rigorous!) upper bound for the degrees.

This would, *in one stroke*, prove, **rigorously**, all the theorems in this article, because it would rigorously justify the (empirically) obvious fact that they belong to the *polynomial ansatz*, and hence discoverable by *undetermined coefficients*, by gathering enough data, and solving a (usually large) system of linear equations, thereby turning the undetermined coefficients of the desired polynomial expressions into *determined* ones. (See below for details.)

Note that in many cases in combinatorics (e.g. [BZ], [E], [Z3]), such arguments are very easy, but in the present case, we don't see it. We hope that one of our readers will!

(s,t)-Core Partitions and Drew Armstrong's Ex-Conjecture

Recall that a *partition* is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $k \geq 0$, called its *number of parts*; $n := \lambda_1 + \ldots + \lambda_k$ is called its *size*, and we say that λ is a *partition of* n.

Also recall that the *Ferrers diagram* (or equivalently, using empty squares rather than dots, *Young diagram*) of a partition λ is obtained by placing, in a *left-justified* way, λ_i dots at the i-th row. For example, the Ferrers diagram of the partition $(5, 4, 2, 1, 1)$ is

$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & \\ * & * & & & \\ * & & & & \\ * & & & & \end{array} \quad .$$

Recall also that the *hook length* of a dot (i, j) in the Ferrers diagram, $1 \leq j \leq \lambda_i$, is the number of dots to its right (in the same row) plus the number of dots below it (in the same column) plus one (for itself), in other words $\lambda_i - i + \lambda'_j - j + 1$, where λ' is the *conjugate partition*, obtained by reversing the roles of rows and columns. (For example if $\lambda = (5, 4, 2, 1, 1)$ as above, then $\lambda' = (5, 3, 2, 2, 1)$).

Here is a table of hook-lengths of the above partition, $(5, 4, 2, 1, 1)$:

$$\begin{array}{ccccc} 9 & 6 & 4 & 3 & 1 \\ 7 & 4 & 1 & 1 & \\ 4 & 1 & & & \\ 2 & & & & \\ 1 & & & & \end{array} \quad .$$

It follows that its set of hook-lengths is $\{1, 2, 3, 4, 6, 7, 9\}$. A partition is called an s-core if none of its hook-lengths is s. For example, the above partition, $(5, 4, 2, 1, 1)$, is a 5-core, and an i-core for all $i \geq 10$.

A partition is a *simultaneous* (s, t)-core partition if it avoids both s and t. For example the above partition, $(5, 4, 2, 1, 1)$, is a $(5, 11)$-core partition (and a $(5, 12)$-core partition, and a $(100, 103)$-core partition etc.).

For a lucid and engaging account, see [AHJ].

As mentioned in [AHJ], Jaclyn Anderson ([A]) very elegantly proved the following.

Theorem 0: If s and t are relatively prime positive integers, then there are **exactly**

$$\frac{(s+t-1)!}{s!t!} \quad ,$$

(s, t)-core partitions.

For example, here are the $(3 + 5 - 1)!/(3!5!) = 7$ $(3, 5)$-core partitions:

$$\{empty, 1, 2, 11, 31, 211, 4211\} \quad .$$

Drew Armstrong ([AHJ], conjecture 2.6) conjectured, what is now the following theorem.

Theorem 1: The *average size* of an (s,t)-core partition is given by the nice polynomial

$$\frac{(s-1)(t-1)(s+t+1)}{24} \quad .$$

For example, the (respective) sizes of the above-mentioned $(3,5)$-core partitions are

$$0,1,2,2,4,4,8 \quad ,$$

hence the average size is

$$\frac{0+1+2+2+4+4+8}{7} = \frac{21}{7} = 3 \quad ,$$

and this agrees with Armstrong's conjecture, since

$$\frac{(3-1)(5-1)(3+5+1)}{24} = 3 \quad .$$

Armstrong's conjecture was recently proved by Paul Johnson ([J]) using a very complicated (but ingenious!) argument (that does much more). Shortly after, and almost *simultaneously* (no pun intended) it was re-proved by Victor Wang [Wan], using another ingenious (and even more complicated) argument, that also does much more, in particular, proving an intriguing conjecture of Tewodros Amdeberhan and Emily Sergel Leven ([AL]). Prior to the full proofs by Johnson and Wang, Richard Stanley and Fabrizio Zanello [SZ] came up with a nice (but rather *ad hoc*) proof of the important special case of $(s,s+1)$-core partitions.

We should also mention the very interesting approach in [YZZ], that proved an important special case (also proved in [AL]) of a conjecture of Amdeberhan and Leven (but not directly related to the subject matter of the present article).

Higher Moments

Recall that the r-th moment about the mean of a random variable, X, is $\mathbf{E}[(X-\mathbf{E}[X])^r]$, where $\mathbf{E}$ is the *expectation* operation.

Theorem 2: If s and t are relatively prime positive integers, then the *variance* (aka as the *second moment (about the mean)*) of the random variable "size of an (s,t)-core partition", is given by the nice polynomial expression:

$$\frac{1}{1440}\, st\,(t-1)\,(s-1)\,(s+t+1)\,(s+t) \quad .$$

For example for $(3,5)$-core partitions it equals

$$\frac{1}{1440}\, 3\cdot 5\,(5-1)\,(3-1)\,(3+5+1)\,(3+5) = 6 \quad ,$$

and indeed (since the average is 3)

$$\frac{(0-3)^2+(1-3)^2+(2-3)^2+(2-3)^2+(4-3)^2+(4-3)^2+(8-3)^2}{7} = 6 \quad .$$

Theorem 3: If s and t are relatively prime positive integers, then the *third moment (about the mean)* of the random variable "size of an (s,t)-core partition" is given by the nice polynomial expression:

$$\frac{1}{60480}\, st\,(t-1)\,(s-1)\,(s+t+1)\,(s+t)\left(2\,s^2t+2\,st^2-3\,s^2-3\,st-3\,t^2-3\right) \quad .$$

For example for $(3,5)$-core partitions it equals

$$\frac{1}{60480}\,3{\cdot}5{\cdot}(5-1){\cdot}(3-1){\cdot}(3+5+1){\cdot}(3+5){\cdot}\left(2\cdot 3^2\cdot 5+2\cdot 3\cdot 5^2-3\cdot 3^2-3\cdot 3\cdot 5-3\cdot 5^2-3\right)=\frac{90}{7} \quad ,$$

and indeed (since the average is 3)

$$\frac{(0-3)^3+(1-3)^3+(2-3)^3+(2-3)^3+(4-3)^3+(4-3)^3+(8-3)^3}{7}=\frac{90}{7} \quad .$$

Theorem 4: If s and t are relatively prime positive integers, then the *fourth moment (about the mean)* of the random variable "size of an (s,t)-core partition" is given by the nice polynomial expression:

$$\frac{1}{4838400}\, st\,(t-1)\,(s-1)\,(s+t+1)\,(s+t)\cdot$$

$$(19\,s^4t^2+38\,s^3t^3+19\,s^2t^4-51\,s^4t-102\,s^3t^2-102\,s^2t^3-51\,st^4+36\,s^4$$

$$+72\,s^3t+108\,s^2t^2+72\,st^3+36\,t^4-33\,s^2t-33\,st^2+36\,s^2+36\,st+36\,t^2+120) \quad .$$

For example for $(3,5)$-core partitions it equals $\frac{726}{7}$, and indeed (since the average is 3)

$$\frac{(0-3)^4+(1-3)^4+(2-3)^4+(2-3)^4+(4-3)^4+(4-3)^4+(8-3)^4}{7}=\frac{726}{7} \quad .$$

Theorem 5: If s and t are relatively prime positive integers, then the *fifth moment (about the mean)* of the random variable "size of an (s,t)-core partition" is given by the nice polynomial expression (in computerish):

```
1/95800320*s*t*(t-1)*(s-1)*(s+t+1)*(s+t)*(46*s**6*t**3+138*s**5*t**4+138*s**4*t**5+46
*s**3*t**6 -211*s**6*t**2 - 633*s**5*t**3-844*s**4*t**4 - 633*s**3*t**5-211*s**2*t**6
+ 333*s**6*t + 999*s**5*t**2 + 1665*s**4*t**3+1665*s**3*t**4 + 999*s**2*t**5+333*s*t**6
- 180*s**6-540*s**5*t - 1283*s**4*t**2-1666*s**3*t**3 - 1283*s**2*t**4 - 540*s*t**5
-180*t**6 + 420*s**4*t + 840*s**3*t**2 + 840*s**2*t**3 + 420*s*t**4-180*s**4 - 360*s**3*t
- 540*s**2*t**2 - 360*s*t**3 - 180*t**4 + 327*s**2*t + 327*s*t**2 - 180*s**2 - 180*s*t
- 180*t**2 - 3780)    .
```

For example for $(3,5)$-core partitions it equals $\frac{2850}{7}$, and indeed (since the average is 3)

$$\frac{(0-3)^5+(1-3)^5+(2-3)^5+(2-3)^5+(4-3)^5+(4-3)^5+(8-3)^5}{7}=\frac{2850}{7} \quad .$$

Theorem 6: If s and t are relatively prime positive integers, then the *sixth moment (about the mean)* of the random variable "size of an (s,t)-core partition" is given by the nice polynomial expression (in computerish):

```
1/4184557977600*s*t*(t-1)*(s-1)*(s+t+1)*(s+t)*(307561*s**8*t**4+1230244*s**7*t**5+
1845366*s**6*t**6+1230244*s**5*t**7+307561*s**4*t**8-2056306*s**8*t**3-8225224*s**7*t**4-
14394142*s**6*t**5-14394142*s**5*t**6-8225224*s**4*t**7-2056306*s**3*t**8+5372061*s**8*t
**2+21488244*s**7*t**3+42976488*s**6*t**4+53720610*s**5*t**5+42976488*s**4*t**6+21488244
*s**3*t**7+5372061*s**2*t**8-6453396*s**8*t-25813584*s**7*t**2-60704054*s**6*t**3-
91764618*s**5*t**4-91764618*s**4*t**5-60704054*s**3*t**6-25813584*s**2*t**7-6453396*s*t
**8+2985120*s**8+11940480*s**7*t+39743142*s**6*t**2+77437746*s**5*t**3+96285048*s**4*t**
4+77437746*s**3*t**5+39743142*s**2*t**6+11940480*s*t**7+2985120*t**8-11104272*s**6*t-
33312816*s**5*t**2-55521360*s**4*t**3-55521360*s**3*t**4-33312816*s**2*t**5-11104272*s*
t**6+2985120*s**6+8955360*s**5*t+23840061*s**4*t**2+32754522*s**3*t**3+23840061*s**2*t**
4+8955360*s*t**5+2985120*t**6-9109476*s**4*t-18218952*s**3*t**2-18218952*s**2*t**3-
9109476*s*t**4+2985120*s**4+5970240*s**3*t+8955360*s**2*t**2+5970240*s*t**3+2985120*t
**4+8664840*s**2*t+8664840*s*t**2-62687520*s**2-62687520*s*t-62687520*t**2+626875200)
.
```

For example for $(3,5)$-core partitions it equals 2346, and indeed (since the average is 3)

$$\frac{(0-3)^6+(1-3)^6+(2-3)^6+(2-3)^6+(4-3)^6+(4-3)^6+(8-3)^6}{7}=\frac{16422}{7}=2346 \quad .$$

The last three theorems regard the special case of $(s,s+1)$-core partitions.

Theorem 7: If s is a positive integer, then the *seventh moment (about the mean)* of the random variable "size of an $(s,s+1)$-core partition" is given by the nice polynomial expression (in computerish):

```
1/149448499200*s**2*(s-1)*(s-2)*(2*s+1)* (124496*s**14 - 527660*s**13 - 127268*s**12+
2133077*s**11 + 1565655*s**10-3928575*s**9-7848989*s**8-3573289*s**7 + 7257797*s**6
+16741975*s**5+16528197*s**4+3583272*s**3 - 67819248*s**2-18541440*s+138620160)
*(s+1)**2    .
```

Theorem 8: If s is a positive integer, then the 8-*th moment (about the mean)* of the random variable "size of an $(s,s+1)$-core partition" is given by the nice polynomial expression (in computerish):

```
1/914624815104000*s**2 * (s-1)*(2*s+1)* (308851624*s**18 - 2759073420*s**17 + 7345195650*s**16
+ 1614779679*s**15 - 27716691813*s**14 - 3203324556*s**13 + 61922226136*s**12 + 52270343442*s**11
- 49025878614*s**10 - 146716496688*s**9 - 153171599682*s**8 - 30342055161*s**7 + 158893451131*s**6
- 165853921776*s**5 + 1073038790016*s**4 + 9260929255680*s**3 - 11293714925568*s**2
- 19188060088320*s + 21924617379840)* (s+1)**2    .
```

Theorem 9: If s is a positive integer, then the *ninth moment (about the mean)* of the random variable "size of an $(s,s+1)$-core partition" is given by the nice polynomial expression (in computerish):

```
1/182467650613248000*s**2*(s-1)*(s-2)*(2*s+1)* (28092743584*s**20 - 284614603048*s**19
+ 908242721124*s**18 - 87722680542*s**17 - 4040707469643*s**16 + 1347179583168*s**15
+ 11350317109273*s**14 + 4824122583716*s**13 - 15816684214230*s**12 - 31535118689736*s**11
- 29475404073738*s**10 + 2671156715274*s**9 + 63014451511513*s**8 + 79700408583680*s**7
+ 45859575725901*s**6 - 377516262865248*s**5 + 6309067352294376*s**4 + 10737857697068736*s**3
- 38301852570773760*s**2 - 26103018295756800*s + 48704747653094400 )* (s+1)**2    .
```

A Crash course in Combinatorial Statistics

Recall ([Z1][Z2]) that given a sequence of combinatorial random variables (e.g. the number of Heads upon tossing a fair coin n times), whose combinatorial generating function is either known explicitly ($C_n(t) = (1+t)^n$ in this trivial case), or only as expressions in t, but for many n, one first finds the discrete *probability generating function* $P_n(t) := C_n(t)/C_n(1)$ (e.g. $P_n(t) = (1+t)^n/2^n$ for coin-tossing), (under the uniform distribution). Of course $P_n(1) = 1$ as it should. To find the *first moment*, av_n (alias average, alias mean, alias expectation) one either derives explicitly $P_n'(1)$ ($n/2$ in this trivial case), or "guesses" in more complicated cases. Next one *centralizes* getting the centralized probability generating function $\tilde{P}_n(t) = P_n(t)/t^{av_n}$, and gets the variance (alias second moment about the mean), $m_2(n)$, by computing (or "guessing") $(t\frac{d}{dt})^2\tilde{P}_n(t)|_{t=1}$, and more generally, the higher moments, $m_k(n)$, are given by $(t\frac{d}{dt})^k\tilde{P}_n(t)|_{t=1}$. From these one derives the *standardized* (scaled) moments $\alpha_k(n) := \frac{m_k(n)}{m_2(n)^{k/2}}$. It is often the case, as $n \to \infty$, that the $\alpha_k(n)$ tend to fixed numbers, in which case we have a *limiting distribution*.

The above is easily extended to the case where the sequence of combinatorial random variables depends on several discrete parameters, like the present case where they depend on the two parameters s and t.

The Wonder of Internet Searches: How we found the (so far conjectured, but absolutely certain) Scaled Limiting Distribution

As mentioned above, more often than not, this limiting distribution is the good-old Gaussian, but this is *definitely* not the case this time. Using our Theorems 2-4, we computed that the limiting *skewness* (standardized third moment) happens to be

$$\lim_{s,t\to\infty} \alpha_3(s,t) = \frac{4}{7}\cdot\sqrt{10} \approx 1.807 \quad ,$$

while the limiting *kurtosis* (standardized fourth moment) turns out to be

$$\lim_{s,t\to\infty} \alpha_4(s,t) = \frac{57}{7} \approx 8.1429 \quad .$$

Being citizens of our time, we *googled*

```
1.807 8.1429 statistics
```

and, *lo and behold*, got the link

`www.aueb.gr/conferences/Crete2015/Papers/Dalla.pdf` ,

(reference [DGP]), that mentioned that these are the skewness and kurtosis of the continuous probability distribution that they call $U_{VS}(1)$ (already mentioned in the introduction), but we will call Z

$$Z := \sum_{k=1}^{\infty} \frac{z_k^2 + \tilde{z}_k^2}{4\pi^2 k^2} \quad ,$$

where z_k and $\tilde{z}_k$ are jointly independent sequences of independent normal random variables.

Liudas Giraitis, one of the authors of [DGP], kindly offered the following interesting information via email.

"VS" stands for "rescaled variance statistic". Characterization of the $U_{VS}(1)$ distribution as the sum of the weighted sum of iid normals was well- known in statistical literature. It was established by Geoffrey S. Watson ([Wat]) in the context of goodness-of-fit tests on a circle, see formula (15) in his paper. Later in [GKLT], the authors show that the VS statistic has the same limit as the limit derived by Watson in the context of goodness-of-fit tests on a circle.

A Human Intermezzo: A Crash Course in Moment Generating Functions

This inspired us to compute higher moments of the continuous probability distribution, Z, as follows.

Recall that the *moment generating function* of a probability distribution X is the (exponential) generating function

$$M_X(t) := \sum_{k=0}^{\infty} \frac{m_k}{k!} t^k \quad .$$

It is well-known and trivial to see that, for any fixed constant, c,

$$M_{cX}(t) = M_X(ct) \quad ,$$

and it is also well-known and easy (but not utterly trivial) to see that if X and Y are *independent* random variables, then

$$M_{X+Y}(t) = M_X(t) M_Y(t) \quad .$$

Hence if $\{X_i\}_{i=1}^{\infty}$ is a sequence of pairwise independent random variables and $\{c_i\}_{i=1}^{\infty}$ is a sequence of positive numbers such that their sum converges, then

$$M_{\sum_{i=1}^{\infty} c_i X_i}(t) = \prod_{i=1}^{\infty} M_{X_i}(c_i t) \quad .$$

If, in addition, the X_i's are *identically distributed*, denoting $M_{X_i}(t)$ by $M(t)$, we have

$$M_{\sum_{i=1}^{\infty} c_i X_i}(t) = \prod_{i=1}^{\infty} M(c_i t) \quad .$$

The famous moments of the standard normal distribution, z, are, as mentioned above, 0 for r odd and $r!/(2^{r/2}(r/2)!)$ for r even. Hence the r-th moment of z^2 is $(2r)!/(2^r r!)$. Hence the (exponential) moment generating function is

$$M_{z^2}(t) \;=\; \sum_{r=0}^{\infty} \frac{(2r)!}{2^r r!} \cdot \frac{t^r}{r!} = \sum_{r=0}^{\infty} \frac{(2r)!}{2^r r!^2} t^r = (1-2t)^{-1/2} \quad .$$

Hence the (exponential) moment generating function of $z^2 + \tilde{z}^2$, where z and $\tilde{z}$ are independent standard normal distributions, is

$$M_{z^2+\tilde{z}^2}(t) = \left((1-2t)^{-1/2}\right)^2 \;=\; (1-2t)^{-1} \quad .$$

Hence the (exponential) moment generating function of the continuous distribution Z is

$$M_Z(t) = \prod_{k=1}^{\infty} (1 - \frac{t}{2\pi^2 k^2})^{-1} \quad .$$

But thanks to good old Leonhard's *iconic* $\sin(\pi x) = \pi x \prod_{k=1}^{\infty} (1 - \frac{x^2}{k^2})$, this equals

$$\frac{\sqrt{t/2}}{\sin \sqrt{t/2}} \quad ,$$

and this can be used to find as many (straight) moments as desired (take the coefficient of t^k in the Maclaurin expansion and multiply by $k!$). From these straight moments one easily computes the *moments about the mean*, using, thanks to the binomial theorem, $\mathbf{E}[(X - m_1)^k] = \sum_{i=0}^{k} \binom{k}{i} (-m_1)^{k-i} m_i$, and from them the standardized moments.

[This is implemented in procedure `VSmoms(N)` in the Maple package `stCore`].

Using this, one of us (SBE) found:

$$\alpha_3 = \frac{4}{7} \cdot \sqrt{10} \quad , \quad \alpha_4 = \frac{57}{7} \quad , \quad \alpha_5 = \frac{820}{77} \cdot \sqrt{10} \quad ,$$

$$\alpha_6 = \frac{1537805}{7007} \quad , \quad \alpha_7 = \frac{466860}{1001} \cdot \sqrt{10} \quad , \quad \alpha_8 = \frac{193032265}{17017} \quad ,$$

$$\alpha_9 = \frac{70231858960}{2263261} \cdot \sqrt{10} \quad ,$$

and these coincide *exactly* (up to the ninth moment) with the limiting (scaled) moments of our combinatorial random variable "size of an (s,t)-core partition", as (s,t) go to infinity, (and $s-t$ is bounded).

How did we derive the above Explicit Expressions for the First Nine Moments?

Let us now briefly describe how we were able to discover the above very deep theorems regarding the first nine moments.

The first step was to use Anderson's bijection, as modified in [AHJ], and to work with (s,t)-**Dyck Paths**, that are two-dimensional walks, using horizontal and vertical positive unit steps, from the origin to the point (s,t) staying *above* the line $s\,y\, -\, t\,x\, =\, 0$.

[Equivalently, the number of ways of tossing a coin $t+s$ times such that you win s dollars if it lands Heads and lose t dollars if it is Tails, breaking even at the end, but never being in debt].

Each such path has associated with it a certain set of positive integers, nicely described in [AHJ], and also important is how many such positive integers there are. Introducing the formal variables q and w, we associate the *weight* $q^s w^k$ to each such path, where s is the sum of the labels and k is the number of such positive labels (see [AHJ] for exactly how to determine them).

The same definitions apply to *partial walks* from $(0,0)$ to (i,j) where we look at the vertical steps $(i,j-1) \rightarrow (i,j)$, and we naturally associate the partial weight due to each such vertical step (assuming (s,t) are fixed) as follows (for the sake of typographical clarity we use $x**y$ for x^y).

$$Wt(i,j)(q,w) := q** \left(\sum_{\{i' \geq i\,;\, s\,j - t\,i' - b > 0\}} s\,j - t\,i' - b \right) \cdot w** \left(\sum_{\{i' \geq i\,;\, s\,j\, -\, t\,i' - b > 0\}} 1 \right) \quad ,$$

then the *weight-enumerator* of the set of partial walks from $(0,0)$ to (i,j) may be computed by the recurrence (we suppress the implied dependence on (q,w))

$$F_{s,t}(i,j) \;=\; F_{s,t}(i-1,j) + Wt(i,j-1) \,\cdot\, F_{s,t}(i,j-1) \quad .$$

We impose the obvious *initial condition* $F_{s,t}(0,0) \,=\, 1$, and *boundary conditions* $F_{s,t}(i,j) \,=\, 0$ if $i < 0$ or $s\,j\, -\, t\,i\, <\, 0$.

Using this *linear recurrence scheme*, one can compute these weight-enumerators very fast.

But $F_{s,t}(i,j)(q,w)$ is but a *stepping stone.* At the end of the day we take $F_{s,t}(s,t)(q,w)$, expand it in terms of powers of w and perform the *umbral substitution*

$$w^k \rightarrow q^{-k(k-1)/2} \quad .$$

(This corresponds to going from the vector of hook-lengths of the first column of the (s,t)-core partition to its shape, see [AHJ]).

All this is implemented in procedure `Fab(a,b,q)` in our Maple package `stCore`.

Using this efficient, *Dynamical Programming*, approach we can *crank out* these weight-enumerators, $F_{s,t}(q)$, for many choices of coprime pairs (s,t), and by repeatedly applying the operator $q\frac{d}{dq}$, and then plugging-in $q = 1$ collect numerical data for the respective moments, and from these, numerical data (for many distinct (s,t)) about the moments about the mean. Then, inspired by Drew Armstrong's ex-conjecture, that expressed the average as a **polynomial**, we assume (without any *a priori* theoretical justification!, see the second challenge) that these are polynomials (i.e. we use the *polynomial ansatz*), and use *undetermined coefficients*, and fit the data with polynomials.

A possible Approach for (most probably ugly) mathematical proofs

The difficulty with the $F_{s,t}(q)$, in their initial definition as weight-enumerators, according to "size of (s,t)-core partitions", is that they are only defined for s and t relatively prime. While the notion of (s,t)-Dyck path makes sense when s and t are not relatively prime, the nice formula that enumerates them, $(s+t-1)!/(s!t!)$, is no longer valid.

A natural approach would be not to be hung up on "nice" formulas, and set-up a general recursion schemes for the weight-enumerators of partial walks, $F_{s,t}(i,j)(q,w)$, but no longer restricted to relatively prime s, t. Then get some recurrence scheme that would enable an (ugly, ad-hoc) proof in the so-called *holonomic ansatz*. But since this approach is only likely to produce ugly proofs, and we are *absolutely certain* that our Theorems 2-9 are correct, we rather not even try, and do not encourage anyone to follow this approach. (But those who insist should start with the easier special case when $t = s+1$.)

But we will be *thrilled* if any of our readers would meet our two challenges stated in the introduction!

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