52 Strange Evaluations by Shalosh B. Ekhad

Theorem 1 :

$$F(-n\,,\,-4n-1/2\,,\,-3n\,,\,-1) \;=\; (64/27)^n \cdot \frac{(3/8)_n(5/8)_n}{(1/3)_n(2/3)_n} \quad .$$

Proof :

$$\frac{-2(296n^3-216n^2k+771n^2+651n-372nk+52nk^2-155k+177-4k^3+45k^2)(3n-k+1)k}{(8n+9-2k)(8n+7-2k)(8n+5-2k)(8n+3-2k)(n-k+1)} \quad \square$$

Theorem 2 :

$$F(-n\,,\,-4n-5/2\,,\,-3n-1\,,\,-1) \;=\; (64/27)^n \cdot \frac{(7/8)_n(9/8)_n}{(2/3)_n(4/3)_n} \quad .$$

Proof :

$$(296n^3+1215n^2-216n^2k+1660n+52nk^2-586nk+756-394k-4k^3+71k^2)\cdot$$

$$\frac{-2(3n-k+2)k}{(8n+13-2k)(8n+11-2k)(8n+9-2k)(8n+7-2k)(n-k+1)} \quad \square$$

Theorem 3 :

$$F(-n\,,\,-3n-1/2\,,\,-4n\,,\,4) \;=\; (-27/16)^n \cdot \frac{(1/3)_n(5/6)_n}{(1/4)_n/(3/4)_n} \quad .$$

Proof :

$$\frac{(344n^3-216n^2k+768n^2+522n-282nk+34nk^2-75k+104+13k^2)(4n-k+1)k}{4(6n+7-2k)(6n+5-2k)(6n+3-2k)(n-k+1)(3n+1)} \quad \square.$$

Theorem 4 :

$$F(-n\,,\,-3n-3/4\,,\,-4n-1\,,\,-8) \;=\; (27/4)^n \cdot \frac{(2/3)_n(7/12)_n}{(1/2)_n(3/4)_n}.$$

Proof :

$$\frac{-(2208n^3-1488n^2k+6272n^2+5762n-2620nk+252nk^2-1082k+1707+167k^2)(4n-k+2)k}{12(12n+15-4k)(12n+11-4k)(12n+7-4k)(n-k+1/(3n+2)} \quad \square$$

Theorem 5 :

$$F(-n\,,\,-3n-3/2\,,\,-4n-2\,,\,-8) \;=\; (27/4)^n \cdot \frac{(5/6)_n(7/6)_n}{(3/4)_n(5/4)_n} \quad .$$

Proof :

$$-(1/24)\frac{(184n^2+492n-124nk+326+21k^2-167k)(4n-k+3)k}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n-k+1)} \quad \square$$

Theorem 6 :

$$F(-n\,,\,-3n-1/4\,,\,-4n\,,\,-8)\;=\;(27/4)^n\;\cdot\;\frac{(1/3)_n(5/12)_n}{(1/2)_n/(1/4)_n}.$$

Proof :

$$\frac{(-1/4)(736n^3-496n^2k+1600n^2+1058n+84nk^2-624nk-149k+200+27k^2)(4n-k+1)k}{(12n+13-4k)(12n+9-4k)(12n+5-4k)(n-k+1)(3n+1)} \quad \square$$

Theorem 7 :

$$F(-n\,,\,-3n-1/2\,,\,-4n\,,\,-8)\;=\;(27/4)^n\;\cdot\;\frac{(1/3)_n(2/3)_n}{(1/4)_n(3/4)_n}.$$

Proof :

$$(-1/32)(2208n^4-1488n^3k+6640n^3+252n^2k^2-2988n^2k+7104n^2+248nk^2-1812nk+3156n+53k^2-321k+478)\cdot$$

$$\frac{(4n-k+1)k}{(6n+7-2k)(6n+5-2k)(6n+3-2k)(n-k+1)(3n+2)(3n+1)} \quad \square$$

Theorem 8 :

$$F(-n\,,\,-3n-5/4\,,\,-4n-1\,,\,-8)\;=\;(27/4)^n\;\cdot\;\frac{(2/3)_n(13/12)_n}{(1/2)_n(5/4)_n}.$$

Proof :

$$\frac{(-1/4)(736n^3-496n^2k+2336n^2+2382n-956nk+84nk^2-414k+767+55k^2)(4n-k+2)k}{(12n+17-4k)(12n+13-4k)(12n+9-4k)(n-k+1)(3n+2)} \quad \square$$

Theorem 9 :

$$F(-n\,,\,-3n-3/2\,,\,-4n-2\,,\,4)\;=\;(-27/16)^n\;\cdot\;\frac{(4/3)_n(5/6)_n}{(3/4)_n(5/4)_n}.$$

Proof :

$$\frac{(1/12)(1032n^3-648n^2k+4024n^2+5202n-1690nk+102nk^2-1103k+2228+135k^2)(4n-k+3)k}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n-k+1)(3n+4)} \quad \square$$

Theorem 10 :

$$F(-n\,,\,-3n-7/4\,,\,-4n-2\,,\,-8)\;=\;(27/4)^n\;\cdot\;\frac{(4/3)_n(11/12)_n}{(3/2)_n(3/4)_n}.$$

Proof :

$$\frac{(-1/12)(2208n^3+9216n^2-1488n^2k+12806n+252nk^2-4112nk+5924-2839k+335k^2)(4n-k+3)k}{(12n+19-4k)(12n+15-4k)(12n+11-4k)(n-k+1)(3n+4)} \qquad \square$$

Theorem 11 : Let ω be a primitive cubic root of unity (i.e. $\omega = (1 \pm \sqrt{3}i)/2$), then

$$F(-n\,,\,-3n-1\,,\,-2n\,,\,\omega) \;=\; (-(3/2)\omega+3/4)^n \cdot \frac{(2/3)_n)}{(1/2)_n} \quad .$$

Proof :

$$(129n^2+289n-2\omega n+160-2\omega-88kn+14k\omega n-101k+19k\omega+13k^2-5\omega k^2)\cdot$$

$$\frac{(-1/129)(-5+13\omega)(2n-k+1)k}{(3n+4-k)(3n+3-k)(3n+2-k)(n-k+1)} \qquad \square$$

Theorem 12 :

$$F(-n\,,\,-3n-1/2\,,\,-n+1/2\,,\,4) \;=\; (-27)^n .$$

Proof :

$$\frac{(1/9)k(2n+1-2k)(84n^2+172n-60nk+12k^2-64k+87)}{(6n+7-2k)(6n+5-2k)(6n+3-2k)(n+1-k)} \qquad \square$$

Theorem 13 :

$$F(-n\,,\,-3n-3/2\,,\,-n+1/2\,,\,4) \;=\; (-27)^n \cdot \frac{(5/6)_n(7/6)_n}{(1/2)_n(3/2)_n} \quad .$$

Proof :

$$\frac{(1/3)(28n^2+76n-20nk+51-28k+4k^2)(2n+1-2k)k}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n+1-k)} \qquad \square$$

Theorem 14 : Let $\alpha = 9 \pm 4\sqrt{5}$, then

$$F(-n\,,\,-3n-1/2\,,\,2n+3/2\,,\,\alpha) \;=\; (128/125+128/125\alpha)^n \cdot \frac{(3/4)_n(5/4)_n}{(4/5)_n(6/5)_n} \quad .$$

Proof :

$$(79794+388827n+706989n^2+566136n^3+167796n^4-4518k-4875nk-51972k^2+11429\alpha nk+$$

$$3754\alpha k+2132\alpha k^2+496k^4\alpha+26472k^3-3696k^4-6816kn^3-6174kn^2-127434k^2n-72852k^2n^2+30768k^3n$$

$$-3336n^3\alpha-9466\alpha n^2-8839\alpha n-2718\alpha+10998k\alpha n^2+1060k^2\alpha n^2-2192k^3\alpha n+3336kn^3\alpha+3058k^2\alpha n-2584k^3\alpha)\cdot$$

$$\frac{-(1/74576)(31\alpha-327)k}{(6n+7-2k)(6n+5-2k)(6n+3-2k)(n-k+1)(4n+3+2k)(\alpha-1)}. \quad \square$$

Theorem 15 : Let $\alpha = 9 \pm 4\sqrt{5}$**, then**

$$F(-n\,,\,-3n-3/2\,,\,2n+5/2\,,\,\alpha) \;=\; (128/125+128/125\alpha)^n \cdot \frac{(5/4)_n(7/4)_n}{(7/5)_n(8/5)_n} \quad .$$

Proof :

$$(384115+1282296n+1575413n^2+845796n^3+167796n^4-86193k-153105nk-77610k^2+26963\alpha nk+$$

$$13383\alpha k+730\alpha k^2+496k^4\alpha+34264k^3-3696k^4-6816kn^3-$$

$$74602kn^2-154458k^2n-72852k^2n^2+30768k^3n-3336n^3\alpha-12126\alpha n^2-14517\alpha n-$$

$$5745\alpha+17114k\alpha n^2+1060k^2\alpha n^2-2192k^3\alpha n+3336kn^3\alpha+1842k^2\alpha n-2984k^3\alpha)\cdot$$

$$\frac{k(-1/74576)(31\alpha-327)}{(6n+9-2k)(6n+7-2k)(6n+5-2k)(n-k+1)(4n+5+2k)(\alpha-1)} \quad \square$$

Theorem 16 : Let $\alpha = 2 \pm \sqrt{3}$**, then**

$$F(-n\,,\,-2n-1/2\,,\,-4n-1\,,\,4\alpha) \;=\; (3/4-3/2\alpha)^n \cdot \frac{(2/3)_n}{(1/2)_n} \quad .$$

Proof :

$$(488n^2+978n-20\alpha n+472-23\alpha+94k^2-25k^2\alpha-420kn+52kn\alpha-422k+72k\alpha)\cdot$$

$$\frac{(1/732)(-68+19\alpha)(4n+2-k)k}{(4n+5-2k)(4n+3-2k)(n-k+1)(3n+2)} \quad \square$$

Theorem 17 :

$$F(-n\,,\,-2n-1/3\,,\,-3n\,,\,9) \;=\; (-4)^n \cdot \frac{(1/2)_n}{(1/3)_n} \quad .$$

Proof :

$$\frac{(1/9)k(3n-k+1)(45n^2-21nk+72n-11k+25)}{(6n+7-3k)(6n+4-3k)(n-k+1)(2n+1)} \quad \square.$$

Theorem 18 :

$$F(-n\,,\,-2n-2/3\,,\,-3n-1\,,\,9) \;=\; (-4)^n \cdot \frac{(5/6)_n}{(2/3)_n} \quad .$$

Proof :

$$\frac{(1/6)(15n+19-7k)(3n-k+2)k}{(6n+8-3k)(6n+5-3k)(n-k+1)} \quad \square$$

Theorem 19 :

$$F(-n\,,\,-n-1/2\,,\,4n+9/2\,,\,1/5)\;=\;(\frac{16384}{15625})^n\;\cdot\;\frac{(9/8)_n(11/8)_n(13/8)_n(15/8)_n}{(6/5)_n(9/5)_n(13/10)_n(17/10)_n}\quad.$$

Proof :

$$(3036n^4+1256n^3k+16960n^3+35275n^2+4780n^2k-1184n^2k^2+6180nk-$$

$$3440nk^2-544nk^3+32445n+2720k+11154-2480k^2-64k^4-800k^3)\cdot$$
$$\frac{(-5/16)k}{(2n+3-2k)(n-k+1)(8n+13+2k)(8n+11+2k)(8n+9+2k)}\quad\Box$$

Theorem 20 :

$$F(-n\,,\,-n+1/2\,,\,4n+3/2\,,\,1/5)\;=\;(\frac{16384}{15625})^n\;\cdot\;\frac{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}{(3/5)_n(2/5)_n(9/10)_n(11/10)_n}\quad.$$

Proof :

$$(3036n^4+1256n^3k+7852n^3+7409n^2+2844n^2k-1184n^2k^2+2252nk-1344nk^2+2993n-$$

$$544nk^3-336k^2+592k+430-352k^3-64k^4)\cdot$$
$$\frac{(-5/16)k}{(2n+1-2k)(n-k+1)(8n+7+2k)(8n+5+2k)(8n+3+2k)}\quad\Box$$

Theorem 21 : Let $\beta\;=\;(3\pm sqrt5)/2$, then

$$F(-n\,,\,2n+1\,,\,-3n\,,\,\beta)\;=\;((25/27)\,+\,(25/27)\beta)^n\;\cdot\;\frac{(2/5)_n(3/5)_n}{(1/3)_n(2/3)_n}\quad.$$

Proof :

$$(1376n+308+2026n^2+958n^3-60k-23k^3\beta+38k^2\beta-91k\beta+139\beta n+87\beta n^2$$

$$-201k\beta n-87k\beta n^2+38k^2\beta n+48k^3+4k^2-122kn-110kn^2+4k^2n+52\beta)\cdot$$
$$\frac{(1/4790)k(3n-k+1)(-78+17\beta)}{(n-k+1)(5n+3)(5n+2)(n+1)(2n+1)}\quad\Box$$

Theorem 22 : Let $\beta\;=\;(3\pm sqrt5)/2$, then

$$F(-n\,,\,2n+2\,,\,-3n-1\,,\,\beta)\;=\;(25/27+25/27\beta)^n\;\cdot\;\frac{(4/5)_n(6/5)_n}{(2/3)_n(4/3)_n}\quad.$$

Proof :

$$(4179n+3463n^2+958n^3+24k-317k\beta n-87k\beta n^2+48k^3+54k^2-109kn-110kn^2+4k^2n+96\beta+$$
$$1674+183\beta n+87\beta n^2+38k^2\beta n-23k^3\beta+34k^2\beta-251k\beta)\cdot$$
$$\frac{(1/4790)k(3n-k+2)(-78+17\beta)}{(n-k+1)(5n+6)(5n+4)(2n+3)(n+1)} \quad \square$$

Theorem 23 : Let ϕ be the Golden ratio $(1+\sqrt{5})/2$ then

$$F(-n\,,\,2n+1\,,\,4n+2\,,\,\phi^{-1}) \;=\; (128/125-64/125\phi^{-1})^n\cdot\frac{(3/4)_n(5/4)_n}{(4/5)_n(6/5)_n} \quad .$$

Proof :

$$(2815n+3661n^2+1558n^3+14k^3\phi^{-1}-148\phi^{-1}n-84\phi^{-1}n^2+579k+183k\phi^{-1}n+$$
$$1567kn+1013kn^2+88k\phi^{-1}+37k^2\phi^{-1}n+34k^2\phi^{-1}+$$
$$270k^2n+84k\phi^{-1}n^2+712+206k^2+39k^3-64\phi^{-1})\cdot$$
$$\frac{(-1/779)k(14\phi^{-1}-25)}{(n-k+1)(4n+4+k)(4n+3+k)(4n+2+k)(-1+\phi^{-1})} \quad \square$$

Theorem 24 : Let ϕ be the Golden ratio $(1+\sqrt{5})/2$ then

$$F(-n\,,\,2n+2\,,\,4n+4\,,\,\phi^{-1}) \;=\; (128/125-64/125\phi^{-1})^n\cdot\frac{(5/4)_n(7/4)_n}{(7/5)_n(8/5)_n} \quad .$$

Proof :

$$(7628n+5998n^2+1558n^3+14k^3\phi^{-1}-178\phi^{-1}n$$
$$-84\phi^{-1}n^2+3188+1762k+309k\phi^{-1}n+2697kn+1013kn^2+253k\phi^{-1}+$$
$$37k^2\phi^{-1}n+79k^2\phi^{-1}+270k^2n+84k\phi^{-1}n^2+387k^2+39k^3-94\phi^{-1})\cdot$$
$$\frac{(-1/779)k(14\phi^{-1}-25)}{(n-k+1)(4n+6+k)(4n+5+k)(4n+4+k)(-1+\phi^{-1})} \quad \square$$

Theorem 25 : Let $\gamma \,=\, 3\pm 2\sqrt{2}$, then

$$F(-2n\,,\,-4n-1/2\,,\,-2n+1/2\,,\,\gamma) \;=\; (-16+96\gamma)^n\frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n} \quad .$$

Proof :

$$(25434+137162n+271600n^2+233440n^3+73472n^4+5216k^2\gamma n^2+10144k^2\gamma n-21148k\gamma n-$$

$$6208k\gamma n^3 - 20304k\gamma n^2 - 1264k^3\gamma n - 142224kn^2 - 58944kn^3 - 111324kn + 14304k^2n^2 +$$
$$22536k^2n - 1776k^3n - 1372k^3 + 8594k^2 + 88k^4 + 4834k^2\gamma - 7067k\gamma - 1204k^3\gamma - 28145k + 3837\gamma +$$
$$104k^4\gamma + 13852\gamma n + 6208\gamma n^3 + 16240\gamma n^2)\cdot$$
$$\frac{(-1/1148)(-89+13\gamma)(4n+1-2k)k}{(8n+9-2k)(8n+7-2k)(8n+5-2k)(8n+3-2k)(2n-k+1)(2n-k+2)(\gamma-1)} \quad \square$$

Theorem 26 :

$$F(-2n\,,\,-4n-1/3\,,\,2/3\,,\,-8)\;=729^n\,.$$

Proof :

$$(314496n^4 + 1101600n^3 + 1416600n^2 + 789644n + 160228 - 1012068kn^2 - 393984kn^3 -$$
$$846054kn - 229356k + 172044k^2n^2 + 290142k^2n + 119151k^2 - 32076k^3n - 26730k^3 + 2187k^4)\cdot$$
$$\frac{(1/729)(-1+3k)k}{(2n-k+1)(2n-k+2)(12n+4-3k)(12n+7-3k)(12n+10-3k)(12n+13-3k)} \quad \square$$

Theorem 27 : Let $\delta \;=\; 5 \pm 3\sqrt{3}$**, then**

$$F(-2n\,,\,-4n-1/3\,,\,2/3\,,\,2\delta)\;=\;(702\delta+135)^n\;\cdot\;\frac{(7/12)_n}{(3/4)_n}\quad.$$

Proof :

$$(2059512 + 12099448n + 766292\delta n + 25671232n^2 + 23410464n^3 + 7767936n^4 + 933000\delta n^2 -$$
$$1239770k\delta n - 404352kn^3\delta + 506562k^2\delta n + 268020k^2\delta n^2 - 62100k^3\delta n - 1257180k\delta n^2$$
$$-2121648k - 5589216kn^3 - 12542136kn^2 - 9090980kn + 1684440k^2n^2 + 2508684k^2n$$
$$-240624k^3n - 56934k^3\delta + 232953k^2\delta - 390660k\delta + 370656n^3\delta$$
$$+205020\delta + 13446k^4 - 178812k^3 + 904542k^2 + 4941k^4\delta)\cdot$$
$$\frac{(1/364122)k(-1+3k)(61\delta-776)}{(12n+13-3k)(12n+10-3k)(12n+7-3k)(12n+4-3k)(2n-k+1)(2n-k+2)2\delta-1)} \quad \square$$

Theorem 28 : Let $\delta \;=\; 5 \pm 3\sqrt{3}$**, then**

$$F(-2n\,,\,-4n-2/3\,,\,4/3\,,\,2\delta) = (702\delta+135)^n\;\cdot\;\frac{(5/12)_n}{(5/4)_n}\quad.$$

Proof :

$$(3710744 + 18504312n + 607916\delta n + 33746032n^2 + 26726016n^3 + 7767936n^4 + 661656\delta n^2 -$$
$$1315958k\delta n - 404352kn^3\delta + 533802k^2\delta n + 268020k^2\delta n^2 - 62100k^3\delta n - 1280676k\delta n^2 -$$

$$3169096k - 5589216kn^3 - 14213784kn^2 - 11770892kn + 1684440k^2n^2 + 2829540k^2n - 240624k^3n$$
$$-60696k^3\delta + 260835k^2\delta - 439228k\delta + 235872n^3\delta + 182668\delta$$
$$+13446k^4 - 200988k^3 + 1159494k^2 + 4941k^4\delta)\cdot$$
$$\frac{(1/364122)k(1+3k)(61\delta - 776)}{(12n+14-3k)(12n+11-3k)(12n+8-3k)(12n+5-3k)(2n-k+1)(2n-k+2)(2\delta-1)} \quad \Box$$

Theorem 29 :

$$F(-2n\,,\,-4n-2/3\,,\,4/3\,,\,-8) \;=\; 729^n \cdot \frac{(5/12)_n(11/12)_n}{(3/4)_n(5/4)_n} \quad .$$

Proof :

$$(34944n^4 - 43776kn^3 + 131104n^3 - 121596kn^2 + 181768n^2 + 19116k^2n^2 -$$
$$110586kn + 110164n + 35046k^2n - 3564k^3n + 15741k^2 + 243k^4 - 32868k + 24564 - 3240k^3)\cdot$$
$$\frac{(1/81)(1+3k)k}{(12n+14-3k)(12n+11-3k)(12n+8-3k)(12n+5-3k)(2n-k+1)(2n-k+2)} \quad \Box$$

Theorem 30 : Let $\alpha \,=\, 2 \pm \sqrt{3}$, then

$$F(-2n\,,\,-3n-1/4\,,\,-4n\,,\,4\alpha) \;=\; (39/2\alpha - 21/4)^n \cdot \frac{(5/12)_n}{(1/4)_n} \quad .$$

Proof :

$$(171978n + 238000n^2 + 104736n^3 - 12496k\alpha n^2 - 32440k\alpha n + 7780k^2\alpha n + 17488\alpha n^2 +$$
$$30900\alpha n + 8407k^2\alpha - 1216k^3\alpha - 18581k\alpha + 39182 + 13262\alpha$$
$$-81808kn^2 + 15490k^2 - 1792k^3 - 43403k - 122224kn + 21112k^2n)\cdot$$
$$\frac{(-1/3273)k(4n-k+1)(397\alpha - 1484)}{(12n+13-4k)(12n+9-4k)(12n+5-4k)(2n-k+1)(2n-k+2)} \quad \Box$$

Theorem 31 :

$$F(-2n\,,\,-3n+1\,,\,-2n+2\,,\,-2) \;=\; 27^n \cdot \frac{(1/2)_n}{(-1/2)_n} \quad .$$

Proof :

$$\frac{(-1/9)k(2n-k-1)(39n^2 + 55n - 21kn - 14k + 3k^2 + 18)}{(3n+2-k)(3n+1-k)(3n-k)(2n+2-k)} \quad \Box$$

Theorem 32 :

$$F(-2n\,,\,-3n\,,\,2n+1/2\,,\,1/4) \;=\; (27/64)^n \cdot \frac{(1/3)_n(2/3)_n}{(1/4)_n(3/4)_n} \quad .$$

Proof :

$$\frac{(-2/3)(74n^3+168n^2-57kn^2-87kn+124n+12k^2n-32k+30-k^3+9k^2)(4n+1-2k)k}{(3n+3-k)(3n+2-k)(3n+1-k)(2n+1-k)(2n+2-k)} \qquad \Box$$

Theorem 33 : Let $\eta \;=\; -7 \pm 4\sqrt{3}$, then

$$F(-2n\,,\,-3n-1/4\,,\,-n+3/4\,,\,\eta) \;=\; (312\eta+24)^n \;\cdot\; \frac{(5/12)_n}{(1/4)_n} \quad .$$

Proof :

$$(485694n+611008n^2+248736n^3-5600k^2\eta n+11504k\eta n^2+23720\eta nk-11880\eta n-$$
$$6512\eta n^2+752k^3\eta+22744k^2+11851\eta k-5432\eta k^2-204544nk-79751k-2416k^3$$
$$+28576k^2n-126928kn^2-5299\eta+124358)\cdot$$
$$\frac{(1/124368)(47\eta+809)(4n+1-4k)k}{(12n+13-4k)(12n+9-4k)(12n+5-4k)(2n-k+1)(2n-k+2)(\eta-1)} \qquad \Box$$

Theorem 34 :

$$F(-2n\,,\,-2n+1/3\,,\,-n+5/6\,,\,-1/8\,) \;=\; (27/16)^n \;\cdot\; \frac{(1/3)_n}{(1/6)_n} \quad .$$

Proof :

$$\frac{(-4/27)(132n^2-60nk+202n+74+9k^2-45k)(6n+1-6k)k}{(6n+5-3k)(6n+2-3k)(2n-k+1)(2n-k+2)} \qquad \Box$$

Theorem 35 :

$$F(-2n\,,\,-2n+1/2\,,\,4n+3/2\,,\,-1/3) \;=\; (4096/6561)^n \;\cdot\; \frac{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}{(1/3)_n/(5/6)_n/(7/12)_n/(13/12)_n} \quad .$$

Proof :

$$(21483+241989n+1106816n^2+2625708n^3+3409136n^4+2299968n^5+631040n^6-$$
$$405096kn^2-723616kn^3-213504kn^5-628608kn^4-161808k^2n^2-175872k^2n^3-72960k^2n^4-10936k^2$$
$$-109724kn+31744k^3n^3+5856k^3-67368k^2n-768k^5+5376k^4n^2+6720k^4n+2624k^4$$
$$+31136k^3n+54144k^3n^2-256k^6-1536k^5n-11388k)\cdot$$
$$\frac{(-3/32)k}{(4n+3-2k)(4n+1-2k)(2n-k+2)(2n-k+1)(8n+7+2k)(8n+5+2k)(8n+3+2k)} \qquad \Box$$

Theorem 36 :

$$F(-2n\,,\,-2n+1/4\,,\,4n+5/4\,,\,1/9)\;=\;(65536/59049)^n\;\cdot\;\frac{(9/16)_n(5/16)_n(13/16)_n(17/16)_n}{(1/3)_n(5/6)_n(13/24)_n(25/24)_n}\quad.$$

Proof :

$$(501813+5630279n+25382756n^2+59064356n^3+75057056n^4+49523456n^5+13285376n^6+1953360kn^2$$
$$-59200kn^3-1363968kn^5-2306048kn^4-9668544k^2n^2-9942016k^2n^3-3846144k^2n^4$$
$$-684064k^2+1280440kn+753664k^3n^3+36608k^3-4189920k^2n$$
$$-16384k^5+368640k^4n^2+523264k^4n+195584k^4+446720k^3n+1058816k^3n^2-16384k^6-49152k^5n+247832k)\cdot$$
$$\frac{(-9/128)k}{(8n+7-4k)(8n+3-4k)(2n-k+2)(2n-k+1)(16n+13+4k)(16n+9+4k)(16n+5+4k)}\quad\Box$$

Theorem 37 :

$$F(-2n\,,\,-2n+1/2\,,\,4n+3/2\,,\,1/9)\;=\;(65536/59049)^n\;\cdot\;\frac{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}{(7/12)_n(5/12)_n(11/12)_n(13/12)_n}\quad.$$

Proof :

$$(693315+7099938n+29906648n^2+66047424n^3+80454656n^4+51184128n^5+13285376n^6$$
$$-3206400kn^2-5387776kn^3-1363968kn^5-4392960kn^4-9505536k^2n^2-$$
$$9778176k^2n^3-3846144k^2n^4-696736k^2-960464kn+753664k^3n^3+218880k^3$$
$$-4165824k^2n-36864k^5+368640k^4n^2+491520k^4n+181760k^4+935936k^3n+1413120k^3n^2$$
$$-16384k^6-49152k^5n-121896k)\cdot$$
$$\frac{(-9/4096)k}{(4n+3-2k)(4n+1-2k)(2n-k+2)(2n-k+1)(8n+7+2k)(8n+5+2k)(8n+3+2k)}\quad\Box$$

Theorem 38 :

$$F(-2n\,,\,-2n-1/4\,,\,4n+11/4\,,\,1/9)\;=\;(65536/59049)^n\;\cdot\;\frac{(11/16)_n(15/16)_n(19/16)_n(23/16)_n}{(4/3)_n(5/6)_n(19/24)_n(31/24)_n}.$$

Proof :

$$(6391980+44661417n+128402584n^2+194585468n^3+163941024n^4+72772864n^5+13285376n^6-18793632kn^2$$
$$-17944128kn^3-1363968kn^5-8184832kn^4-17754048k^2n^2-13460480k^2n^3$$
$$-3846144k^2n^4-2275184k^2-9616576kn+753664k^3n^3+846464k^3-10406208k^2n-69632k^5$$

$+368640k^4n^2+644096k^4n+281088k^4+2419968k^3n+2332672k^3n^2-16384k^6-49152k^5n-1949232k)\cdot$

$$\frac{(-9/128)k}{(8n+9-4k)(8n+5-4k)(2n-k+2)(2n-k+1)(16n+19+4k)(16n+15+4k)(16n+11+4k)} \qquad \square$$

Theorem 39 : Let $\zeta = -1 \pm \sqrt{2}$, then

$$F(-2n\,,\,1/2\,,\,-4n\,,\,2\zeta) \;=\; \frac{(3/8)_n\,(5/8)_n}{(1/4)_n\,(3/4)_n} \quad .$$

Proof :

$$\frac{(-1/2)(3+\zeta)(6\zeta n-2n+4\zeta-3k^2-k^2\zeta+8kn+6k)(4n-k+1)k}{(2n-k+2)(2n-k+1)(8n+5)(8n+3)} \qquad \square$$

Theorem 40 :

$$F(-2n\,,\,1/2\,,\,4n+5/2\,,\,1/4) \;=\; \frac{(7/8)_n\,(5/8)_n\,(9/8)_n\,(11/8)_n}{(3/4)_n^2\,(5/4)_n^2} \quad .$$

Proof :

$$\frac{(-4/3)(1+2k)(256n^3+640n^2-24k^2n+550n-4k^3-24k^2+k+162)k}{(2n-k+2)(2n-k+1)(8n+9+2k)(8n+7+2k)(8n+5+2k)} \qquad \square$$

Theorem 41 :

$$F(-2n\,,\,1/2\,,\,4n+3/2\,,\,1/4) \;=\; \frac{(3/8)_n\,(7/8)_n\,(5/8)_n\,(9/8)_n}{(7/12)_n\,(5/12)_n\,(11/12)_n\,(13/12)_n} \quad .$$

Proof :

$$\frac{(-4/81)(-1+6k)(2304n^3+4608n^2-216k^2n+144kn+3214n-36k^3-144k^2+117k+770)k}{(2n-k+2)(2n-k+1)(8n+7+2k)(8n+5+2k)(8n+3+2k)} \qquad \square$$

Theorem 42 : Let ω be a primitive cubic root of unity (i.e. $\omega = (1 \pm \sqrt{3}i)/2$), then

$$F(-2n\,,\,2n+1\,,\,-4n\,,\,\omega) \;=\; ((27/16)\omega)^n \,\cdot\, \frac{(1/3)_n\,(5/6)_n}{(1/4)_n\,(3/4)_n} \quad .$$

Proof :

$(2660+14784n+30156n^2+26880n^3+8848n^4+308\omega+1316\omega n-11k^4\omega-16k^4-1936kn^3\omega-4852kn^2\omega-4086kn\omega$

$+204k^3+1234k^2\omega n+708k^2\omega n^2-526k^2-6052kn^2-2816kn^3+236k^3n-4234kn+2k^3\omega-1148\omega k+537\omega k^2-780k^2n^2$

$$+24k^3\omega n-1322k^2n+1792\omega n^2+784\omega n^3-966k)\cdot$$

$$\frac{(1/6636)k(4n-k+1)(-27+11\omega)}{(2n+1)(2n-k+2)(2n-k+1)(6n+5)(3n+1)(n+1)} \quad \square$$

Theorem 43 : Let ϕ be the Golden ratio $(1+\sqrt{5})/2$ then

$$F(-2n\,,\,2n+1\,,\,-3n+1/2\,,\,\phi) \;=\; ((100/27)\phi+25/9)^n \cdot \frac{(3/10)_n(7/10)_n}{(1/6)_n(5/6)_n} \quad .$$

Proof :

$$(1678n+2708n^2+1448n^3+136k^2\phi n-264k\phi n^2-364k\phi n-118k\phi+324\phi n$$

$$+264\phi n^2-26k^3\phi+96k^2\phi+343+96\phi-267k-19k^2+24k^3-42k^2n-536kn^2-750kn)\cdot$$

$$\frac{(2/905)k(6n+1-2k)(-43+16\phi)}{(2n+1)(2n-k+2)(2n-k+1)(10n+7)(10n+3)} \quad \square$$

Theorem 44 : Let ϕ be the Golden ratio $(1+\sqrt{5})/2$ then

$$F(-2n\,,\,2n+1\,,\,-3n\,,\,\phi) \;=\; (100/27\phi+25/9)^n \cdot \frac{(2/5)_n(3/5)_n}{(1/3)_n(2/3)_n}.$$

Proof :

$$(1334n+1716n^2+724n^3-270k\phi n+68k^2\phi n-132k\phi n^2-125k\phi+214\phi n+132\phi n^2$$

$$-13k^3\phi+68k^2\phi+82\phi-177k-21k^2+12k^3-21k^2n-268kn^2-433kn+342)\cdot$$

$$\frac{(1/905)k(3n-k+1)(-43+16\phi)}{(2n-k+2)(2n-k+1)(5n+3)(5n+2)(n+1)} \quad \square$$

Theorem 45 : Let $\zeta_1 \;=\; 1\pm\sqrt{2}$, then

$$F(-2n\,,\,2n+1\,,\,-2n+1/2\,,\,(1/2)\zeta_1) \;=\; (4+8\zeta_1)^n \cdot \frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n} \quad .$$

Proof :

$$(56n^2+72n+8\zeta_1 n+24+5\zeta_1-6k^2-16kn-12k+4k^2\zeta_1-8\zeta_1 kn-6\zeta_1 k)\cdot$$

$$\frac{(2/7)k(4n+1-2k)(-4+\zeta_1)}{(2n-k+2)(2n-k+1)(8n+5)(8n+3)}\square$$

Theorem 46 : Let $\alpha \;=\; 2\pm\sqrt{3}$, then

$$F(-2n\,,\,2n+1\,,\,-n+3/4\,,\,(1/4)\alpha) = (3-6\alpha)^n \cdot \frac{(5/12)_n}{(1/4)_n} \quad .$$

Proof :

$$\frac{(1/3)(8n+5+\alpha-2k\alpha+2k)(-5+\alpha)(4n+1-4k)k}{(2n-k+2)(2n-k+1)(12n+5)} \quad \square$$

Theorem 47 :

$$F(-2n\,,\,3n+1\,,\,-2n+2\,,\,2/5) \;=\; (125/27)^n \cdot \frac{(1/2)_n}{(-1/2)_n} \quad .$$

Proof :

$$\frac{(-1/25)(147n^2+33nk+179n+3k^2+23k+50)(2n-k-1)k}{n(2n+2-k)(3n+2)(3n+1)} \quad \square$$

Theorem 48 :

$$F(-2n\,,\,4n+1\,,\,-4n+1/2\,,\,2) \;=\; 16^n \cdot \frac{(5/16)_n(3/16)_n(13/16)_n(11/16)_n}{(3/8)_n(1/8)_n(7/8)_n(5/8)_n} \quad .$$

Proof :

$$(20886+229992n+1021408n^2+2335744n^3+2908160n^4$$

$$+1875968n^5+491520n^6-852992kn^4-1028864kn^3-278528kn^5$$

$$-611776kn^2+86016k^2n^3+30720k^2n^4+43152k^2n-5344k^3n-5760k^3n^2$$

$$-2048k^3n^3-1583k^3+91168k^2n^2$$

$$+278k^4+480k^4n+128k^4n^2-68k^5+8k^6-64k^5n+7628k^2-179848nk-21029k)\cdot$$

$$\frac{(-1/4)(8n+1-2k)k}{(2n-k+2)(2n-k+1)(16n+11)(16n+13)(16n+3)(16n+5)(4n+3)(4n+1)} \quad \square$$

Theorem 49 :

$$F(-2n\,,\,4n+2\,,\,-4n-1/2\,,\,2) \;=\; 16^n \cdot \frac{(9/16)_n(7/16)_n(15/16)_n(17/16)_n}{(3/8)_n(7/8)_n(5/8)_n(9/8)_n} \quad .$$

Proof :

$(268890+1832424n+5143712n^2+7618816n^3+6283264n^4+2736128n^5+491520n^6-1309696kn^4$

$-2438912kn^3-278528kn^5-2249600kn^2+125952k^2n^3+30720k^2n^4+134312k^2n-10400k^3n-8064k^3n^2-2048k^3n^3-$

$4313k^3+194848k^2n^2+518k^4+640k^4n+128k^4n^2-92k^5+8k^6-64k^5n+34640k^2-1028112nk-186251k)\cdot$

$$\frac{(-1/4)(8n+3-2k)k}{(2n-k+2)(2n-k+1)(16n+17)(16n+15)(16n+7)(16n+9)(4n+5)(4n+3)} \quad \square$$

Theorem 50 :

$$F(-2n\,,\,4n+1\,,\,-2n+2\,,\,1/3) \;=\; (81/16)^n \;\cdot\; \frac{(1/2)_n}{(-1/2)_n}.$$

Proof :

$$(1040n^3+1812n^2+264kn^2+982n+36k^2n+342kn+24k^2+162+103k+2k^3)\cdot$$

$$\frac{(-1/54)(2n-k-1)k}{n(2n-k+2)(4n+3)(2n+1)(4n+1)} \quad \square$$

Theorem 51 : Let $\zeta_2 \;=\; 1 \pm \sqrt{3}/3$**, then**

$$F(-2n\,,\,4n+2\,,\,4/3\,,\,(2/3)\zeta_2) \;=\; (-9+18\zeta_2)^n \;\cdot\; \frac{(5/12)_n}{(5/4)_n} \quad .$$

Proof :

$$(352n^2+432n+168\zeta_2 n+116+138\zeta_2-2k^2+9\zeta_2 k^2-$$

$$288\zeta_2 nk+152nk+126k-237\zeta_2 k)\cdot$$

$$\frac{(-1/88)(-16+9\zeta_2)(1+3k)k}{(2n-k+2)(2n-k+1)(12n+5)(4n+3)} \quad \square$$

Theorem 52 : Let $\zeta_2 \;=\; 1 \pm \sqrt{3}/3$**, then**

$$F(-2n\,,\,4n+1\,,\,2/3\,,\,(2/3)\zeta_2) \;=\; (-9+18\zeta_2)^n \;\cdot\; \frac{(7/12)_n}{(3/4)_n} \quad .$$

Proof :

$$(352n^2+264n+264\zeta_2 n+20+174\zeta_2-2k^2+9\zeta_2 k^2-288\zeta_2 nk+152nk+102k-195\zeta_2 k)\cdot$$

$$\frac{(-1/88)(-16+9\zeta_2)(-1+3k)k}{(2n-k+2)(2n-k+1)(12n+7)(4n+1)} \quad \square$$