The Number of Same-Sex Marriages in a Perfectly Bisexual Population is Asymptotically Normal

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Theorem: Consider a population of $2n$ men and $2n$ women where every individual is equally attracted to either sex and chooses his or her mate according to other criteria. Also assume that everyone gets married. Then the expectation of the random variable "Number of same-sex marriages" is

$$\frac{2n\,(2\,n-1)}{4\,n-1}$$

that asymptotically is

$$n-\frac{1}{4}-\frac{1}{16}\,n^{-1}-\frac{1}{64}\,n^{-2}-\frac{1}{256}\,n^{-3}-\frac{1}{1024}\,n^{-4}-\frac{1}{4096}\,n^{-5}-\frac{1}{16384}\,n^{-6}-\frac{1}{65536}\,n^{-7}-\frac{1}{262144}\,n^{-8}+O\left(n^{-9}\right) \quad ,$$

and its variance is

$$8\,\frac{n^2\left(1-4\,n+4\,n^2\right)}{-3+28\,n-80\,n^2+64\,n^3} \quad ,$$

that asymptotically is

$$\frac{1}{2}\,n+\frac{1}{8}+\frac{1}{16}\,n^{-1}+\frac{3}{64}\,n^{-2}+\frac{19}{512}\,n^{-3}+\frac{59}{2048}\,n^{-4}+\frac{45}{2048}\,n^{-5}+\frac{17}{1024}\,n^{-6}+\frac{1637}{131072}\,n^{-7}+\frac{4917}{524288}\,n^{-8}+O\left(n^{-9}\right) \quad .$$

Furthermore, this random variable is asymptotically normal.

Semi-Rigorous Proof: The probability generating function is (why?)

$$P_n(x) \;=\; \frac{1}{(4n)!/((2n)!2^{2n})}\sum_{k=0}^{n}\binom{2n}{2k}^2(2n-2k)!((2k)!/(k!2^k))^2x^{2k} \quad . \qquad (PGF)$$

By repeatedly applying the operation $f(x)\to xf'(x)$, plugging in $x=1$, for $n=1$ to $n=200$, and fitting the data by a rational function of n, one gets in turn the mean, variance, and higher moments, from which one easily gets the *moments about the mean* and the *normalized moments* (α coefficients).

It turned out that the mean and variance are indeed as stated by the theorem. As for the higher moments, we will show that the first 14 normalized moments tend (as $n\to\infty$) to those of the normal distribution, and the readers can go on as far as they please.

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Accompanied by Maple package `SameSexMarriages` downloadable from `http://www.math.rutgers.edu/~zeilberg/tokhniot/SameSexMarriages` .
More (computer-generated!) detailed output can be gotten from the front of this article `http://www.math.rutgers.edu/~/mamarim/mamarimhtml/ssm.html` .

The normalized third moment (about the mean) is

$$1/2\,\sqrt{2}\sqrt{\frac{-3+4\,n}{n^2\,(25-140\,n+276\,n^2-224\,n^3+64\,n^4)}}\quad ,$$

the asymptotics of its square is:

$$1/32\,n^{-5}+\frac{11}{128}\,n^{-6}+\frac{85}{512}\,n^{-7}+\frac{571}{2048}\,n^{-8}+\frac{3569}{8192}\,n^{-9}+O\left(n^{-10}\right)\quad .$$

Note that it goes to 0, as it should, since the odd moments of the normal distribution are all 0.

The normalized fourth moment about the mean (alias *kurtosis*) is

$$1/2\,\frac{3-100\,n+650\,n^2-1896\,n^3+2632\,n^4-1664\,n^5+384\,n^6}{n^2\,(35-188\,n+348\,n^2-256\,n^3+64\,n^4)}\quad .$$

Its asymptotics is:

$$3-n^{-1}+1/4\,n^{-2}+\frac{7}{16}\,n^{-3}+\frac{57}{64}\,n^{-4}+\frac{431}{256}\,n^{-5}+\frac{3137}{1024}\,n^{-6}+\frac{22431}{4096}\,n^{-7}+\frac{159025}{16384}\,n^{-8}+\frac{1122319}{65536}\,n^{-9}+O\left(n^{-10}\right)\quad .$$

Note that it converges to $4!/(2^2\cdot 2!)=3$, as it should, this being the fourth moment of the normal distribution.

The normalized fifth moment about the mean can be viewed in `http://www.math.rutgers.edu/~zeilberg/tokhniot/oSameSexMarriages1`, and the asymptotics of its square is:

$$\frac{25}{8}\,n^{-5}+\frac{315}{32}\,n^{-6}+\frac{3501}{128}\,n^{-7}+\frac{37067}{512}\,n^{-8}+\frac{383273}{2048}\,n^{-9}+O\left(n^{-10}\right)\quad .$$

Note that it goes to 0, as it should, since the odd moments of the normal distribution are all 0.

The normalized sixth moment about the mean can be viewed in `http://www.math.rutgers.edu/~zeilberg/tokhniot/oSameSexMarriages1` . Its asymptotics is:

$$15-15\,n^{-1}+\frac{31}{4}\,n^{-2}+\frac{73}{16}\,n^{-3}+\frac{839}{64}\,n^{-4}+\frac{8401}{256}\,n^{-5}+\frac{86191}{1024}\,n^{-6}+\frac{903617}{4096}\,n^{-7}+\frac{9635359}{16384}\,n^{-8}+\frac{103978545}{65536}\,n^{-9}+O\left(n\right.$$

Note that it converges to $6!/(2^3\cdot 3!)=15$, as it should, this being the sixth moment of the normal distribution.

Etc. etc. (See `http://www.math.rutgers.edu/~zeilberg/tokhniot/oSameSexMarriages1` for evidence up to the 14^{th} moment, and you are welcome to modify `http://www.math.rutgers.edu/~zeilberg/tokhniot/inSameSexMarriages1` and run it on your computer, if you want more evidence.) $\square$

Comments

1. It should be routine, once we found the average and variance as above (that even humans should be able to do easily), to derive a *local limit law*, by using Stirling's asymptotic formula. However, for some reason, Maple refuses to take the appropriate limit. (It works for the binomial distribution).

2. Another *automatic*, this time *fully rigorous* proof, can also be gotten using an extension of the methods in [Z1], using the Maple package
`http://www.math.rutgers.edu/~zeilberg/tokhniot/CLT`. However, since the polynomials $P_n(x)$ are not closed-form, but rather satisfy a second-order linear recurrence (easily derived via the Zeilberger algorithm) one should be able to do it (but `CLT` has to be slightly extended).

3. The present approach follows [Z2] and the Maple package accompanying it:
`http://www.math.rutgers.edu/~zeilberg/tokhniot/HISTABRUT`.

4. Not directly related to the present article, but nevertheless highly recommended, are references [AZ1], [Z3], [Z4], and [Z5].

References

[AZ1] Moa Apagodu and Doron Zeilberger, *Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory*, Advances in Applied Mathematics **37** (2006) 139-152.

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[Z2] Doron Zeilberger, *HISTABRUT: A Maple Package for Symbol-Crunching in Probability theory*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, Aug. 25, 2010. (Journal Main page: `http://www.math.rutgers.edu/ zeilberg/pj.html`, direct url of article:
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[Z4] Doron Zeilberger, *Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials*, Amer. Math. Monthly **96**(1989), 590-602.

[Z5] Doron Zeilberger, *The Umbral Transfer-Matrix Method. I. Foundations* Journal of Combinatorial Theory, Series A, **91** (2000), 451-463.