How To Generate As Many Somos-Like Miracles as You Wish

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Dedicated to the GURU of Non-Linear Recurrence Equations, Gerry LADAS

Dedication

It is a great pleasure to dedicate this article to Gerry Ladas, who along with Saber Elaydi, made *Difference Equations* such an active and exciting field.

Somos Sequences

In the late 1980s, Michael Somos came up with an amazing set of conjectures, featuring non-linear recurrences that always yield integers. The most famous one was Somos-4: [S]`http://oeis.org/A006720`. Define a sequence $\{a(n)\}_{n=1}^{\infty}$ by $a(1) = 1$, $a(2) = 1$, $a(3) = 1$, $a(4) = 1$, and for $n \geq 5$,

$$a(n) = \frac{a(n-1)a(n-3) + a(n-2)^2}{a(n-4)} \quad .$$

Prove that $a(n)$ is an integer for every $n \geq 1$. This is indeed amazing, since, starting at $n = 9$, you divide by an integer larger than 1, and *a-priori* you are only guaranteed to get *rational numbers*, but surprise-surprise, you *always* (seem) to get integers.

This took quite some effort to prove *rigorously*. Of course, confirming it for $n \leq 300$, say, already constitutes a (very convincing!) *empirical* proof, but mathematicians do not care about *truth* per-se, they only care about playing their (sometimes fun but often very dull) *artificial* **game** called *proving* ("rigorously", using *logical* deduction).

And indeed, such a mathematical (elegant!) proof was given by Janice Malouf[M]. Other people gave other proofs, see the references given at [S] `http://oeis.org/A006720`.

Of course, Michael Somos is a **honest** person. He first encountered this sequence in some context, and then made his conjecture. But suppose that you are **dishonest**, and want to amaze your friends (or challenge your foes) with such *Somos Phenomena*, coming up with non-linear recurrences where you know *beforehand*, that you are *guaranteed* to get integers, and even have a fully rigorous proof of that fact.

Then you must follow the advise of Carl Gustav Jacob Jacobi who taught us that in order to make life simpler,

"*man muss immer umkehren*" ,

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in other words, work backwards! Instead of trying to play darts, by placing the target on the wall, and then aiming from a distance of twenty meters, and most probably missing it, one **first** throws the dart, and *then* draws the bull's eye!

Consider the linear recurrence sequence defined by

$$f(1)=1\,,\; f(2)=3\,,\; f(n)=3f(n-1)-f(n-2) \quad (n \geq 3) \quad .$$

Obviously (by induction on n) they are **all** integers.

Consider, on the other hand, the following sequence

$$b(1)=1\,,\; b(2)=3\,,\; b(n)=\frac{b(n-1)^2-1}{b(n-2)} \quad (n \geq 3) \quad ,$$

now it is not at all immediately obvious that the $b(n)$ are always integers.

Suppose that one asked the question "prove that the $b(n)$ are always integers". The **best** way to prove it would be to generate the first few terms, go to Neil Sloane's OEIS[S], and find out that $b(n) = A001519(n)$. In other words it is the same as the above sequence $f(n)$, at least for the first 30 terms.

So far it is only a "conjecture", but if you know about the C-finite ansatz ([K][KP][Z1][Z2]), then knowing that $f(n) = b(n)$ for $1 \leq n \leq 4$ implies that it is true for *all* $n \geq 1$, thereby yielding an 'empirical', **yet fully rigorous**, proof!

It is easy to see that C-finite sequences, i.e. sequences that satisfy **linear** recurrence equations with *constant coefficients*, also satisfy (many!) **non-linear** recurrences. But more surprisingly, the same is true for any **subsequence** where the **places** are either **polynomials** (e.g. n^2) or **exponential** (e.g. 2^n), or even **exponential polynomial** (e.g. $n^2 2^n + n^4 3^n$). For example, if $F(n)$ are the Fibonacci numbers defined by $F(0)=0, F(1)=1, F(n)=F(n-1)+F(n-2) (n \geq 2)$, then the sequence

$$a(n) := F(n^2 2^n + n^3 5^n + 1)$$

satisfies some (very complicated!) non-linear recurrence equation, i.e. there exists a positive integer r (the order) and a polynomial $P(x_0, \dots, x_r)$ such that

$$P(a(n), \dots, a(n+r)) = 0 \quad (for \;\; all \;\; n \geq 1).$$

Furthermore, by possibly increasing r, one can demand that the degree in x_r is 1, so one can express $a(n+r)$ as a *rational function* of $a(n), \dots, a(n+r-1)$ and get a genuine non-linear recurrence that *very surprisingly* (if you don't know how it was found!) yields integers.

Of course the same is true for *any* sequence that satisfies a linear recurrence equation with constant coefficients.

So let's state the main theorem in its intimidating full generality.

Main Theorem: Let $f(n)$ be a sequence satisfying a linear recurrence equation with constant coefficients

$$f(n) = \sum_{i=1}^{d} c_i f(n-i) \quad ,$$

for some constants $c_1, \ldots, c_d$, and with some given initial conditions $f(1), \ldots, f(d)$. Let $p(n)$ be of the form, for some specific integer m,

$$p(n) := POL(n, 2^n, 3^n, \ldots, m^n) \quad ,$$

where POL is some polynomial of m variables. Then there exists an integer r and a polynomial $P(x_0, \ldots, x_r)$, of degree 1 in x_r such that the subsequence
$a(n) := f(p(n))$ satisfies the non-linear recurrence

$$P(a(n), \ldots, a(n+r)) = 0 \quad (n \geq 1) \quad .$$

The formal proof of this theorem would be very boring, but its informal proof, at least *generically*, is not too bad. Using Binet's formula, one can express $f(n)$ "explicitly" as a linear combination of α^n (and possibly $n\alpha^n, n^2\alpha^n$ etc. in case of multiple roots) where the α's are roots of the equation

$$1 = \sum_{i=1}^{d} c_i \alpha^{-i} \quad .$$

By using the initial conditions, one can find the coefficients in that linear combination of the α^n's, that are also certain expressions in the α's. Then, since $\alpha^{(n+1)^2} = \alpha^{n^2}(\alpha^n)^2\alpha$, $\alpha^{2^{n+1}} = (\alpha^{2^n})^2$ etc. one can introduce *auxiliary variables*, $\{\alpha^n, \alpha^{n^2}, \alpha^{n^3}, \ldots, \alpha^{2^n}, \alpha^{3^n}, \ldots\}$, (finitely many!), and recall that the α's all satisfy the algebraic equation above (and enjoy nice relations amongst themselves, e.g. that their sum is c_1 and their product is $(-1)^{d-1}c_d$).

Now one can express $f(p(n))$ in terms of these auxiliary variables, and by *elimination* (in principle using the Buchberger algorithm), one can get, for some r, a pure polynomial relation, of the desired form $P(a(n), \ldots, a(n+r)) = 0$, and even demand that the degree in $a(n+r)$ is one, so that one can express $a(n+r)$ as a rational function of $a(n), \ldots, a(n+r-1)$.

In practice, it is better to use *undetermined coefficients*. Fixing r and d, write a *generic* polynomial $P(x_0, \ldots, x_r)$ of total degree d (and degree 1 in x_r), plug-in the expressions in terms of the auxiliary variables, get a huge polynomial in these variables, set up all the coefficients (w.r.t. the auxiliary variables) of the numerator to 0, and solve the large system of linear equations where the unknowns are the "undetermined coefficients", thereby hopefully making them determined. If the only solution is the trivial zero solution, don't give up! Just raise d and/or r.

Now if we take the coefficients of the recurrence $c_1, \ldots, c_d$ to be *integers*, and the initial conditions $f(1), \ldots, f(d)$ to be integers as well, then the sequence $f(n)$ obviously consists of only integers. If, in addition, the polynomial POL has all positive integer coefficients, then we are guaranteed, *a priori*, that the terms of the subsequence $a(n) := f(p(n))$ are all integers!

For example, if our sequence is $f(n) = F_{2n}$, i.e. $f(1) = 1$, $f(2) = 3$,
$f(n) = 3f(n-1) - f(n-2) \quad (n \geq 3)$, considered above, then the Binet formula is

$$f(n) = \frac{\alpha^n - \alpha^{-n}}{\alpha - \alpha^{-1}} \quad ,$$

where α is a root of the quadratic equation $\alpha^2 - 3\alpha + 1 = 0$. Taking $p(n) = n^2$ we get

$$a(n) = \frac{\alpha^{n^2} - \alpha^{-n^2}}{\alpha - \alpha^{-1}} \quad .$$

Now

$$a(n+1) = \frac{\alpha^{(n+1)^2} - \alpha^{-(n+1)^2}}{\alpha - \alpha^{-1}} = \frac{\alpha^{n^2}(\alpha^n)^2\alpha - \alpha^{-n^2}(\alpha^{-n})^2\alpha^{-1}}{\alpha - \alpha^{-1}}$$

$$a(n+2) = \frac{\alpha^{(n+2)^2} - \alpha^{-(n+2)^2}}{\alpha - \alpha^{-1}} = \frac{\alpha^{n^2}(\alpha^n)^4\alpha^4 - \alpha^{-n^2}(\alpha^{-n})^4\alpha^{-4}}{\alpha - \alpha^{-1}} \quad .$$

So you have expressions for $a(n)$,$a(n+1)$, $a(n+2)$, as rational functions in the auxiliary variables $A := \alpha^n$, $B := \alpha^{n^2}$ (and α, but α satisfies $\alpha^2 - 3\alpha + 1 = 0$). So generically we should have a second-order polynomial relation $P(a(n), a(n+1), a(n+2)) = 0$. But it is not of degree one in $a(n+2)$. If we allow higher-order recurrences, it turns out that we have a fairly nice non-linear recurrence of order 5. Here is one such example.

Prop. Let $a(n)$ be defined by

$$a(1) = 1 \quad , \quad a(2) = 21 \quad , \quad a(3) = 2584 \quad , \quad a(4) = 2178309 \quad , \quad a(5) = 12586269025$$

and for $n \geq 6$, by the recurrence

$$a(n) = (\,2303\,a(n-4)a(n-3)a(n-1) + 2255\,a(n-3)^2a(n-2) \;+\; 329a(n-4)a(n-1)^2$$

$$-\,15792\,a(n-4)a(n-2)^2 \;+\; 329\,a(n-4)a(n-3)^2 - 2303\,a(n-4)^2a(n-2) + 441\,a(n-2)$$

$$-\,a(n-2)^3 - 2961\,a(n-4) - a(n-5)a(n-2)a(n-1) + 329\,a(n-5)a(n-3)a(n-2)\,)/(\,48\,a(n-4)a(n-2)\,) \quad ,$$

then $a(n)$ are all integers!

For many more such propositions, see:
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oNesSomos3` .

The much sparser subsequence, $a(n) := f(2^n)$ satisfies a much simpler (second-order!) non-linear recurrence:

$$a(n) = \frac{a\,(n-1)\left(4 + 5\,a\,(n-1)^2\right)}{2 + 5\,a\,(n-2)^2}$$

subject to the initial conditions

$$a(1) = 3 \quad , \quad a(2) = 21 \quad .$$

The Maple package NesSomos

Everything is implemented in the Maple package `NesSomos` (written by DZ) available directly from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/NesSomos` ,

and that is linked to from the "front" of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/somos.html` , where one can also find three long webbbooks generated by SBE.

To get a list of the main procedures, type `ezra();`, and for help with a specific procedure, type `ezra(ProcedureName);`. Let's just list the more important procedures.

• `FindREL(LPols,Vars,x,d)` is a very general procedure that inputs a list of rational functions `LPols` in the list of variables `Vars`, a symbol, `x`, and a positive integer `d`, and outputs a polynomial of degree $\leq d$, let's call it $P(x[1], \dots, x[r])$ such that

$$P(LPols[1], \dots, LPols[r]) = 0 \quad .$$

For example,

$$FindREL([p^2 - q^2, 2pq, p^2 + q^2], [p, q], x, 2);$$

yields

$${x_3}^2 - {x_2}^2 - {x_1}^2 = 0 \quad .$$

If nothing is found, then it returns $FAIL$ and one has to increase d. Of course in order to *guarantee* that there is such a good d, we must insist that the length of `LPols` is larger than the length of `Vars`.

• `FindInREL(LPols,Vars,x,d)` is exactly as above, but, in addition, the output should have degree one in $x[r]$ (where r is the length of `LPols`)). This is needed for the application to generating Somos-like sequences.

• `FindSMpG(c,p,n,Maxr,d,a);` inputs a positive integer `c`, a polynomial expression `p` in the discrete variable `n`, positive integers `Maxr` and `d`, and a symbol `a`, and outputs a non-linear recurrence, of order `Maxr` and degree $\leq d$ satisfied by the sequence $a(n) := f(p(n))$, where $f(n)$ is the sequence satisfying $f(n) = cf(n-1) - f(n-2)$ and $f(0) = 0, f(1) = 1$. If it fails it returns `FAIL`, and the human has to try again with a larger `d` and/or `Marx`. The first amazing example above was gotten from

```
FindSMpG(3,n**2,n,5,6,a);
```

• `FindSMeG(c,e,n,Maxr,d,a);` inputs a positive integer `c`, a positive integer `e`, positive integers `Maxr` and `d` and a symbol `a`, and outputs a non-linear recurrence of order `Maxr` and degree $\leq d$

satisfied by the sequence $f(e^n)$, where $f(n)$ is the sequence satisfying $f(n) = cf(n-1) - f(n-2)$ and $f(0) = 0, f(1) = 1$. If it fails it returns `FAIL`.

The second amazing example above was gotten from:

```
FindSMeG(3,2,n,3,3,a);
```

Webbbooks

For many amazing Somos-like miracles see the three webbooks

- `http://www.math.rutgers.edu/~zeilberg/tokhniot/oNesSomos1`
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/oNesSomos3`
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/oNesSomos4`

Enjoy!

Using `NesSomos` you can find many more Somos-like *miracles* on your own, impressing your friends and challenging your enemies.

Laurent Phenomenon

Analogous things can be done to generate sequences that have the so-called Laurent phenomenon, and even, *a priori*, the much stronger property of *positivity*.

Encore

By looking at the output of
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oNesSomos1` , we see a pattern that can be summarized as follows.

Proposition: Let c be a positive integer (a formal symbol), and define a sequence by

$$a(0) = 1 \quad , \quad a(1) = c \quad ,$$

and for $n \geq 2$,

$$a(n) = \frac{a(n-1)\ \left(4 + (c-2)(c+2)a(n-1)^2\right)}{2\ +\ (c-2)(c+2)a(n-2)^2} \quad .$$

Then for all $n \geq 1$, $a(n)$ are integers (polynomials in c with integer coefficients).

Proof: Consider the sequence defined by $f(0) = 0, f(1) = 1$ and for $n \geq 2$ by

$$f(n) = cf(n-1) - f(n-2) \quad .$$

Define $b(n) := f(2^n)$. It is routine to prove that

$$b(n) = \frac{b(n-1)(4 + (c-2)(c+2)b(n-1)^2)}{2 + (c-2)(c+2)b(n-2)^2} \quad ,$$

and, of course, $b(0) = 1$ and $b(1) = c$. Hence, by induction, $b(n) = a(n)$ for all $n \geq 1$. Since the $f(n)$ are obviously integers (polynomials in c with integer coefficients), so are $a(n)$. $\square$

References

[K] Manuel Kauers, *SumCracker: A package for manipulating symbolic sums and related objects*, Journal of Symbolic Computation **41** (2006), 1039-1057,
`http://www.risc.uni-linz.ac.at/people/mkauers/publications/kauers06h.pdf` .

[KP] Manuel Kauers and Peter Paule, *"The Concrete Tetrahedron"*, Springer, 2011.

[M] J. L. Malouf, "An integer sequence from a rational recursion", Discr. Math. **110** (1992), 257-261.

[S] Neil Sloane, "*The On-Line Encyclopedia of Integer Sequences*", `http://oeis.org/` .

[Z1] Doron Zeilberger, *An Enquiry Concerning Human (and Computer!) [Mathematical] Understanding* in: C.S. Calude ,ed., "Randomness and Complexity, from Leibniz to Chaitin", World Scientific, Singapore, Oct. 2007,
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/enquiry.html` .

[Z2] Doron Zeilberger, *The C-finite Ansatz*, to appear in the Ramanujan Journal.
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cfinite.html` .