Identities in character tables of S_n

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1 New section added by DZ (INCOMPLETE!)

It should be possible (hopefully) to prove the following theorem.

Theorem 1.1. *Let μ be any partition, and let $|\mu|$ be the sum of its parts. Then, for $n \geq |\mu|$, we have*

$$\sum_{\lambda \vdash n} (\chi^\lambda(\mu, 1^{n-|\mu|}))^2 = C(\mu)(n-|\mu|)! \quad ,$$

where

$$C(\mu) := \sum_{\lambda \vdash |\mu|} (\chi^\lambda(\mu))^2 \quad ,$$

Sketch of Proof Recall that f_λ, the number of Standard Young Tableaux of shape λ, alias $\chi^\lambda(1^n)$, satisfies two famous recurrences

$$f_\lambda = \sum_{\lambda^-} f_{\lambda^-} \quad , \qquad (GoingDown)$$

where the sum is over all partitions λ^- obtained by removing one cell from λ (legally). This recurrence is trivial combinatorially, but it should not be too hard to prove it directly, either from representation theory, or via the fact at the very last line of p. 114 of Macdonald's Second Ed., that $\chi^\lambda(\rho)$ is the coefficient of $x^{\lambda+\delta}$ in $a_\delta p_\rho$.
It also follows (combinatorially from Robinson-Schenstead, but it is not too hard to prove it directly, once again from the above fact), that, if $n = |\lambda|$, then

$$(n+1) f_\lambda = \sum_{\lambda^+} f_{\lambda_+} \quad , \qquad (GoingUp)$$

where the sum is over all partitions λ^+ obtained by adding one cell to λ (legally).
Let

$$A_n := \sum_{\lambda \vdash n} f_\lambda^2$$

Recall that one way to prove that $A_n = n!$ (w/o using R-S) is to note that, because of the Going Down Formula

$$A_n = \sum_{\lambda \vdash n} f_\lambda^2 = \sum_{\lambda \vdash n, \mu \vdash n-1} f_\lambda f_\mu$$

where the sum is over all pairs λ, μ such that $\lambda - \mu$ is one legal cell. But this is the same, thanks to the Going Up formula, to nA_{n-1}. Indeed:

$$nA_{n-1} = \sum_{\mu \vdash n-1} n f_\mu^2 = \sum_{\mu \vdash n-1} (n f_\mu) f_\mu \sum_{\lambda \vdash n, \mu \vdash n-1} f_\lambda f_\mu \quad ,$$

with the same summation set.
To prove the theorem we note that the Going Down formula seems to hold for all μ ending in 1

$$\chi^\lambda(\mu) = \sum_{\lambda^-} \chi^{\lambda^-}(\mu') \quad , \qquad (GoingDownGeneral)$$

where μ' is μ with the last part (that must be a 1, by assumption) removed.
An analog of the Going Up formula seems to hold (see the Maple package `SnCharacterTableMiracles`) for any μ, we have

$$(N_1(\mu) + 1)\ \chi^\lambda(\mu) = \sum_{\lambda^+} \chi^{\lambda^+}(\mu_1) \quad , \qquad (GoingUpGeneral)$$

where $N_1(\mu)$ is the number of ones in μ, and μ_1 is the partition obtained from μ by appending a 1.
The proof of the theorem now follows the same way.

References

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[3] I.G. Macdonald, Symmetric Functions

[4] The Online Encyclopedia of Integer Sequences.

[5] R. Stanley, EC II