Spelling Out The First Part of Roger Myerson's Article "The Autocrat's Credibility..."

Amer. Poly. Sci. Rev. Vol. 102(1), Feb. 2008, 125-139

Tamar ZEILBERGER

General Set-Up

• War occurs, on the average, every $\frac{1}{\lambda}$ years, and it is assumed that the probability of a war occurring in a short interval of length ϵ is $\epsilon\lambda$, so (as is well-known), the *waiting time* T until the next war is an **exponential** random variable with parameter λ. Recall that the *probability density function* is $\lambda e^{-\lambda t}$, hence the **cumulative prob. function is**, i.e. that a war would happen in $\leq T$ years is (recall, from calc 2 that $\int e^{kt}\,dt = \frac{e^{kt}}{k}$ for any constant k)

$$\int_0^T \lambda e^{-\lambda t}\,dt = \lambda \frac{e^{-\lambda t}}{-\lambda}|_0^T = \lambda \frac{e^{-\lambda t}}{-\lambda}|_0^T = -e^{-\lambda T} - (-e^0) = 1 - e^{-\lambda T} \quad .$$

Note that if λ is small, it is unlikely that there is a war soon. For example, if $\lambda = .05$ then the probability of there being a war in ≤ 3 years is $1 - e^{-0.05 \cdot 3} = 1 - e^{-0.15} = 0.1392920236$, so the probability that there is going to be peace for the next three years is $e^{-0.15} = 0.8607079764$

• The discount rate (compounded continuously, as usual) is δ, so an amount of y dollars promised to be paid in t years is worth today only $e^{-\delta t} y$ dollars.

• The ruler's gross income is R, but if he wants to survive, he has to use some of it to pay his captains. Of course, being a greedy bastard, he would like to pay them the least amount that they are willing to accept.

• The cost of a captain participating in a war supporting the leader is c

• There is a parameter s (usually taken ≤ 2), such that the **probability** that n captains winning against m captains of the enemy, let's call it $p(n|m)$, is given by the (artificial) formula

$$p(n|m) = \frac{n^s}{m^s + n^s} \quad .$$

Note that if $m = n$ then the probability is exactly $\frac{1}{2}$ for *every* s, but if, say, $n = 2m$ then the probability of winning is

$$p(n|m) = \frac{(2m)^s}{m^s + (2m)^s} = \frac{2^s}{1 + 2^s} \quad ,$$

so if $s = 1$ it is $\frac{2}{3}$ but if $s = 2$ it is $\frac{2^2}{1+2^2} = \frac{4}{5}$, so the higher the s the more favorable is it to the stronger force.

The Content of the Lemma

Suppose someone promises you y dollars when a certain event will occur, but you don't know when it will happen. You only know the prob. distribution of T. Then the *expected* values is $yE[e^{-\delta T}$, for the *exponential* distribution it is (using the formula above)

$$\frac{y}{\delta+\lambda} \quad .$$

The Actual Value for a Supporting Captain of a promise of a stream of y dollars

Like in the Rubinstein Bargaining Model, we assume an *infinitely repeated game* .

Let $U(n, y|m)$ be the *actual* value of an offer of a stream of y dollars from the ruler then it satisfies the following *algebraic equation* (that Meyerson calls "recursive").

$$U(n, y|m) = \frac{y}{\delta+\lambda} + \frac{\lambda}{\delta+\lambda} \cdot [\, p(n|m) \cdot U(n, y|m) - c\,] \qquad (RecursiveEquationForU)$$

Important Note: There is a serious typo in the paper it is written

$$U(n, y|m) = \frac{y}{\delta+\lambda} + \frac{\lambda}{\delta+\lambda} \cdot p(n|m) \cdot [U(n, y|m) - c]$$

In other words the "[" in front of $U(n, y|m)$ should move to the left, between $\frac{\lambda}{\delta+\lambda}$ and $p(n|m)$.

Let's explain every part of this equation.

- $\frac{y}{\delta+\lambda}$ is the **expected** gain of the income stream **until** the next war.

- $p(n|m)$ is the probability of winning the war, hence the expected utility, right after the next war is

$$p(n|m) \cdot U(n, y|m) + (1 - p(n|m)) \cdot 0 - c = p(n|m) \cdot U(n, y|m) \;-\; c \quad .$$

(Note that the cost is fixed, whether there is a win or a loss).

But, in *today's dollars*, the expected value of this, according to the lemma, is obtained by multiplying the latter by $\frac{\lambda}{\delta+\lambda}$.

Adding all these ups we get $(RecursiveEquationForU)$.

Comment: Everything is according to expectation, so it is tacitly assumed that the captains and the ruler are *risk neutral.*

Now using simple *high-school algebra*, we can *solve* the equation $(RecursiveEquationForU)$.

First open-up parantheses on the right side of the equation, in order to get

$$U(n,y|m) = \frac{y}{\delta+\lambda} + \frac{\lambda}{\delta+\lambda} \cdot p(n|m) \cdot U(n,y|m) - \frac{\lambda c}{\delta+\lambda} \quad .$$

Moving the term involving $U(n,y|m)$ to the left hand side, we get

$$U(n,y|m) - \frac{\lambda}{\delta+\lambda} \cdot p(n|m) \cdot U(n,y|m) = \frac{y}{\delta+\lambda} - \frac{\lambda c}{\delta+\lambda} \quad .$$

Simplifying we get

$$U(n,y|m)(1 - \frac{\lambda}{\delta+\lambda} \cdot p(n|m)) \;=\; \frac{y-\lambda c}{\delta+\lambda}$$

that leads to the **explicit expression** for $U(n,y|m)$

$$U(n,y|m) = \frac{y - \lambda\, c}{\delta + \lambda - \lambda\, p(n|m)} \quad . \qquad (ExplicitUatPeace)$$

This is the **expected payoff** for the captain at time of peace.

But, on the Eve of battle it is

$$-c + p(n|m)\, U(n,y|m) \qquad (ExplicitUatEveOfWar)$$

because, the cost for the captain of going to war is c, and since right after the war, it is peace again, and his payoff is then $U(n,y|m)$, but only in the event of winning, and it is 0 if they lose, so the expected payoff at the eve of war is indeed $-c + p(n|m)U(n,y|m)$. Plugging-in the expression for $U(n,y|m)$ and doing a simple high-school algebra manipulation, we get that his expected payoff at the eve of battle is

$$\frac{p(n|m)y - c(\lambda+\delta)}{\lambda + \delta - \lambda p(n|m)} \quad .$$

The captain, not being a sucker, will accept the payment stream y only if this quantity is **positive**, or, at least **non-negative**. Since the denominator is obviously positive, we need to set the numerator to be ≥ 0, getting the condition

$$p(n|m)\, y - c\,(\lambda+\delta) \geq 0 \quad .$$

Dividing by $p(n|m)$, we have

$$y - c\frac{\lambda+\delta}{p(m,n)} \geq 0 \quad .$$

Hence

$$y \geq \frac{c\,(\lambda+\delta)}{p(n|m)} \quad .$$

Hence the "minimum wage", let's call it $Y(m|n)$, that the captain will accept is

$$Y(m|n) \;:=\; \frac{c\,(\lambda+\delta)}{p(n|m)} \quad .$$

The Actual Revenue for the Ruler Who Wants to Survive

Let $V(n, y|m)$ be the *expected discounted value* for the ruler with the above scenario (n captains supporting him, m captains fighting against him, and he is offering a stream of y to each captain.

Analogous to Equation ($RecursiveEquationForU$), we have

$$V(n, y|m) = \frac{R - ny}{\delta+\lambda} + \frac{\lambda}{\delta+\lambda} \cdot p(n|m) \cdot V(n, y|m) \quad . \qquad (RecursiveEquationForV)$$

(Because before discounting, the ruler's revenue is $R - ny$ (since there are n captains, each of them getting y). $\frac{R-ny}{\delta+\lambda}$ is the expected discounted value of that, $p(n|m) \cdot V(n, y|m)$ is the expected gain after the next war, and one has to multiply this by $\frac{\lambda}{\delta+\lambda}$ to allow for discounting (to convert is today's dollars).

Using high-school algebra, one solves for $V(n, y|m)$ and gets

$$V(n, y|m) = \frac{R - ny}{\delta + \lambda - \lambda p(n|m)} \qquad (ExplicitEquationForV)$$

This is the expected (discounted) revenue for the leader at time of *peace*. The expected revenue at the *eve of war* is $p(n|m) \cdot V(n, y|m) + (1 - p(n|m)) \cdot 0 = p(n|m) \cdot V(n, y|m) \quad .$

Let's call it $W(n, y|m)$, so we have

Important Fact: the expected pay-off for the leader at the eve of war is

$$W(n, y|m) := \frac{p(n|m)\,(R - ny)}{\delta + \lambda - \lambda\, p(n|m)} \quad .$$

Now the **important question** is whether the leader's promise to pay his captains is credible?

Absolute Leader

Since there is no communication between the captains (so a captain only knows about himself being cheated), the captains can trust the leader only of they know that the ruler has no incentive to cheat any of them. If he would try to reduce his force to $k < n$ captains, then he would only do it if his pay-off then, $V(k, y|m)$ would be larger. Hence the captains are safe if we have the

Feasibility condition for Absolute Ruler

Definition a force of n captains is *feasible for an absolute ruler* against a force of m captains (with payoff $y \geq Y(n|m)$ to captains) if and only if for **every** $k < n$, we have

$$V(k, y|m) \leq V(n, y|m) \quad .$$

In other words, the ruler has no incentive to reduce his force (and hence having to pay less, since there is a trade-off, with a smaller force, his probability of winning the war is less, so he does not come out ahead).

Numerical Example

Myerson gives an example much later, but we will spell out already for this concept, using his parameters (p. 130)

$$R = 90 \quad , \quad \delta = 0.05 \quad , \quad \lambda = 0.2 \quad , \quad c = 5 \quad , \quad s = 1.5 \quad .$$

n=7 and m=10

Let's see whether a force of 7 captains is feasible against a force of 10 captains. With the above parameters, we have

$$Y(n|m) := 5 \frac{(0.2 + 0.05)}{n^{1.5}/(m^{1.5} + n^{1.5})} \quad .$$

So the (minimum payment) is

$$Y(7|10) := 5 \frac{(0.2 + 0.05)}{7^{1.5}/(10^{1.5} + 7^{1.5})} = 3.384336802$$

putting $y = 3.384336802$ in the formula for $V(n, y|m)$, we have

$$V(n, 3.384336802|m) = \frac{90 - n \cdot 3.384336802}{0.05 + 0.2 - 0.2(n^{1.5}/(m^{1.5} + n^{1.5}))}$$

We get

$$V(7, 3.384336802|10) = \frac{90 - 7 \cdot 3.384336802}{0.05 + 0.2 - 0.2(7^{1.5}/(10^{1.5} + 7^{1.5}))} = 376.4806430 \quad .$$

Now let's test every k between 1 and 6.

$$V(6, 3.384336802|10) = \frac{90 - 6 \cdot 3.384336802}{0.05 + 0.2 - 0.2(6^{1.5}/(10^{1.5} + 6^{1.5}))} = 373.6114707 \quad ,$$

since this is **less** the ruler is not tempted to reduce the force to 6.

Taking k all the way to 0 we get

$$V(5, 3.384336802|10) = 369.5317677 \quad , \quad V(4, 3.384336802|10) = 364.7693444 \quad ,$$

$$V(3, 3.384336802|10) = 360.0367405 \quad , \quad V(2, 3.384336802|10) = 356.3288459 \quad ,$$

$$V(1, 3.384336802|10) = 355.1724560 \quad , \quad V(0, 3.384336802|10) = 360.0000000 \quad .$$

Since all these values are **less** than 376.4806430, we just showed that a force of 7 captains is feasible against a force of 10 captains, with the above parameters.

n=5 and m=10

Let's see whether a force of 5 captains is feasible against a force of 10 captains. With the above parameters, we have

$$Y(n|m) := 5\frac{(0.2 + 0.05)}{n^{1.5}/(m^{1.5} + n^{1.5})} \quad .$$

So the (minimum payment) is

$$Y(5|10) := 5\frac{(0.2 + 0.05)}{5^{1.5}/(10^{1.5} + 5^{1.5})} = 4.785533905$$

putting $y = 4.785533905$ in the formula for $V(n, y|m)$, we have

$$V(5, 4.785533905|10) = \frac{90 - 5 \cdot 4.785533905}{0.05 + 0.2 - 0.2(5^{1.5}/(10^{1.5} + 5^{1.5}))} = 334.1049222 \quad .$$

Now let's test every k between 1 and 4.

$$V(4, 4.785533905|10) = \frac{90 - 4 \cdot 4.785533905}{0.05 + 0.2 - 0.2(4^{1.5}/(10^{1.5} + 4^{1.5}))} = 338.0313891 \quad ,$$

since this is **more** the ruler is **tempted** to reduce the force to 4, hence the force of 5 captains is **not** feasible against 10 captains.

By plugging-in values, I found that the **feasibility range** for $m = 10$ (with Myerson's example parameters) for $m = 10$ is $6 \leq n \leq 20$, so starting with $n = 21$ once again it is not feasible.

This section concludes with Proposition 1 that says that it is always beneficial for the ruler to abandon absolutism and proceed to a *weak court* (and later, to a *strong court.* So far the notion of *equilibrium* did not come up, and will only come up much later in this very deep and complicated paper, but I have described Myerson's model and described the framework and many of the key-concepts, in particular "feasibility" of n vs. n captains, that leads to the transition from absolute leader to a leader with a *weak court*, but that's a different story.