The Revenge of the Plain Mathematician:
How I Answered Two Questions by Two Fancy Jerusalem Mathematicians In One Day

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Like all mathematicians who grew up in Israel, I was always in awe of the members of the faculty of the Hebrew University at Jerusalem, roughly the Israeli analog of Princeton. Hence I felt a huge personal satisfaction, when, on the very same day (yesterday, Oct. 4, 1999), I was able to answer, *on the spot*, two questions posed by two *distinct* Jerusalem mathematicians, that furthermore, were utterly trivial to me, and showed me that at least I know something that these fancy guys do not seem to know. I was also amused that, roughly, I am the average of these two, at least in age and in piety.

The first question came from the famous Knotter Dror Bar-Natan (15 years my junior), who asked me to prove the identity:

$$\frac{(q)_{a+b}}{(q)_a(q)_b} = \sum_{c=0}^{b} \frac{q^{ac}}{(q^{-1})_c(q)_{b-c}} \quad ,$$

where $(q)_n := (1-q)(1-q^2)...(1-q^n)$. After a few minutes of blabbing that these identities are all shaloshable (using my Maple package qEKHAD), and him retorting that he meant a *real* proof, not a *bogus* (computer-generated) one, I realized that it would be a waste of Shalosh's precious time to give this trifle to it. All one has to do is set $y := -q^a$ in Cauchy's celebrated (and trivial) q-binomial theorem (e.g. [W])

$$\prod_{k=1}^{b}(1+yq^k) = \sum_{c=0}^{b} y^c q^{c(c+1)/2} \frac{(q)_b}{(q)_c(q)_{b-c}} \quad ,$$

and do some trivial manipulations. $\square$

The second question was posed by the famous Riemann-Surfacer, Herschel Farkas (11 years my senior), later the same day, in a Temple University math colloquium talk. He asked to give an 'elementary' proof of the following identity, that he and collaborators derived using fancy, theta function methods: If k is odd (for some reason Herschel said 'prime', but he probably meant odd, since it is true for all odd k)

$$\sum_{r=0}^{(k-3)/2} \tan^2\left(\frac{(2r+1)\pi}{2k}\right) = (k-1)(k-2)/6 \quad . \qquad (Herschel)$$

I noticed right away, that this is (using $\tan^2 = \sec^2 - 1$), $-(k-1)/2$ plus the sum of the reciprocals of the roots of the $(k-1)/2$-degree polynomial $p(x)$, where $p(x^2) = T_k(x)/x$, $T_k(x)$ being the

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venerable Chebychev polynomial (see. e.g. [W]) (when k is odd): $x(1-((k+1)(k-1)/6)x^2+...)$. Hence $p(x)=1-((k+1)(k-1)/6)x+...$, and the sum of the reciprocals of its roots is $(k+1)(k-1)/6$, hence the left side of $(Herschel)$ equals $(k+1)(k-1)/6-(k-1)/2=(k-1)(k-2)/6$. $\square$.

Moral: You don't have to be a Zeilberger to solve the above two 'problems', they are both trivial, once you know the relevant facts. Hence it is good to assemble as many facts as possible from *all over mathematics.* A painless, indeed extremely enjoyable, way to increase your 'mathematical facts power' is to browse, at least an hour a day, in Eric Weisstein's[W] magnificent opus, either in its paper version (if your age is >60), or the web version (if your age is <40), or both (if you are somewhere in between, like me).

References

[W] Eric W. Weisstein, *CRC Concise Encylopedia of Mathematics*, web version:
`http://www.astro.virginia.edu/~eww6n/#TreasureTroves` .