Two Hankel Determinants dear to Volodia Retakh

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Dedicated to Vladimir Retakh (b. May 20, 1948) on his 70th birthday

Let $B_s := \frac{(2s)!}{s!^2}$, then

$$\det\left(B_{r+i+j}\right)_{0\le i,j\le n-1} =$$

$$\left(\frac{(2\,r)!}{(r!)^2}\right)^n \prod_{n_1=1}^{n}\left(\prod_{n_2=2}^{n_1} 4\,\frac{(n_2-1)\,(2\,n_2-3)\,(2\,n_2-3+2\,r)\,(n_2-2+r)}{(2\,n_2-2+r)\,(2\,n_2-4+r)\,(2\,n_2-3+r)^2}\right) \quad .$$

Let $C_s := \frac{(2s)!}{s!(s+1)!}$, then

$$\det\left(C_{r+i+j}\right)_{0\le i,j\le n-1} =$$

$$\left(\frac{(2\,r)!}{r!\;(r+1)!}\right)^n \prod_{n_1=1}^{n}\left(\prod_{n_2=2}^{n_1} 4\,\frac{(n_2-1)\,(2\,n_2-1)\,(n_2-1+r)\,(2\,n_2-3+2\,r)}{(2\,n_2-1+r)\,(2\,n_2-3+r)\,(2\,n_2-2+r)^2}\right)$$

Proofs

First Proof of both determinants (by SBE). Go to the Maple package

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/CLD` ,

and type

`EvalHpaper((2*r)!/(r!*r!),n,r,4);` ,

and

`EvalHpaper((2*r)!/(r!*(r+1)!),n,r,4);` ,

respectively, getting, *ab initio*, the above expressions, together with proofs, using Dodgson's rule and induction. See [Z].

Sketch of the Second Proof of the First determinant (by DZ)

We use the fact that

$$\frac{(2r)!}{r!^2} = ConstantTerm_x\left(\frac{1}{x}+2+x\right)^r \quad .$$

Let $f(x) = \frac{1}{x} + 2 + x$. The determinant is the Constant term, in $x_1, \dots, x_n$, of

$$\det\left(f(x_i)^{r+i+j-2}\right)_{1\le i,j\le n} \quad .$$

$$= ConstantTerm_{x_1,\dots,x_n} \prod_{i=1}^{n} f(x_i)^r \;\det\left(f(x_i)^{i+j-2}\right)_{1\le i,j\le n} \quad .$$

By Vandermonde (also dear to Volodia Retakh!), this equals

$$ConstantTerm_{x_1,\dots,x_n} \prod_{i=1}^{n} f(x_i)^r \prod_{i=1}^{n} f(x_i)^{i-1} \prod_{1 \le i < j \le n} (f(x_i) - f(x_j)) \quad .$$

Symmetrizing, and using Vandermonde one more time, this equals

$$\frac{1}{n!} CT_{x_1,\dots,x_n} \prod_{i=1}^{n} f(x_i)^r \prod_{1 \le i < j \le n} (f(x_i) - f(x_j))^2 \quad .$$

Now, *lo and behold*, this is a special case $k_1 = r$, $k_2 = 1$, $k_3 = 0$, of the BC_n Macdonald Constant-Term Identity (see, e.g., [FW], top of p. 501).

Sketch of the Second Proof of the Second determinant (by DZ)

We use the fact that

$$\frac{(2r)!}{r!(r+1)!} = ConstantTerm_x (1-x)(\frac{1}{x} + 2 + x)^r \quad .$$

The determinant is the constant term, in $x_1, \dots, x_n$, of

$$\prod_{i=1}^{n} (1 - x_i) \ \det\left(f(x_i)^{r+i+j-2}\right)_{1 \le i,j \le n} \quad .$$

This equals

$$ConstanTerm_{x_1,\dots,x_n} \prod_{i=1}^{n} (1 - x_i) \prod_{i=1}^{n} f(x_i)^r \ \det\left(f(x_i)^{i+j-2}\right)_{1 \le i,j \le n} \quad .$$

By Vandermonde, this equals

$$ConstantTerm_{x_1,\dots,x_n} \prod_{i=1}^{n} (1 - x_i) \prod_{i=1}^{n} f(x_i)^r \prod_{i=1}^{n} f(x_i)^{i-1} \prod_{1 \le i < j \le n} (f(x_i) - f(x_j)) \quad .$$

Symmetrizing this equals

$$\frac{1}{n!} ConstantTerm_{x_1,\dots,x_n} \prod_{i=1}^{n} (1 - x_i) \prod_{i=1}^{n} f(x_i)^r \prod_{1 \le i < j \le n} (f(x_i) - f(x_j))^2 \quad .$$

Now, *lo and behold*, this is a special case $k_1 = r - \frac{1}{2}$, $k_2 = \frac{1}{2}$, $k_3 = 1$, of the BC_n Macdonald Constant-Term Identity mentioned above ([FW], top of p. 501).

Challenges for Volodia

In the beautiful article [BR], the authors prove a non-commutative analog, using quasi-determinants, of the special cases $r = 0$ and $r = 1$ of the second determinant (where the determinant happens to be 1, in the commutative case). What is the non-commutative analog, for general r, of the second determinant? The first determinant?

References

[BR] Arkady Berenstein and Vladimir Retakh, *Noncommutative Catalan Numbers*,
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[FW] Peter J. Forrester and S. Ole Warnaar , *The importance of the Selberg integral*, Bull. Amer. Math. Soc. **45** (2008), 489-534
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[Z] Doron Zeilberger, *Lieber Opa Paul, Ich Bin Auch Ein Experimental Scientist!*, Adv. Applied Math. **31** (2003), 532-543,
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