Proof of a Conjecture of Amitai Regev about Three-Rowed Young Tableaux (and much more!)

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Consider lattice paths in the three-dimensional cubic lattice, with unit positive steps, that always stay in the region $x \geq y \geq z$. Let $A_{b_1,b_2,b_3}(n)$ be the number of such walks of length $n-(b_1+b_2+b_3)$ that start at $[b_1,b_2,b_3]$. For example, $A_{0,0,0}(n)$ is the number of 3D-ballot paths (trivially isomorphic to $\leq$ 3-rowed Standard Young Tableaux with n cells).

In 1981 Amitai Regev[R1] (see also [S]) famously proved that $A_{0,0,0}(n)$ is given by the *Motzkin numbers*: $M(n) := \sum_{k=0}^{n/2} \frac{n!}{k!(k+1)!(n-2k)!}$. A quarter-century later[R2], he conjectured (essentially) the following statement:

Regev's Conjecture: $A_{2,1,0}(n) = M(n-1) - M(n-3)$.

In this note we prove this conjecture. More generally, we present an algorithm to (automatically!) *conjecture* and then, immediately, **rigorously** (and automatically!) prove such statements for *any* given starting point $[b_1,b_2,b_3]$. It turns out that for all the starting points we looked at ($b_3 \leq b_2 \leq b_1 \leq 20$), there are similar expressions as linear combination of negative shifts of the Motzkin sequence $M(n)$ (with **constant** coefficients).

Note that any such statement, in any *dimension*, once conjectured, is decidable via Wilf-Zeilberger theory and the *holonomic ansatz*, at least in principle. In practice, however, because of the multi-sums (or multi-integrals), it becomes unwieldy. We do know, however, that for *any* such conjecture, there exists a (computable!) number N_0 (depending on the conjecture, of course) such that if we verify our conjecture for $n \leq N_0$, then it is true for ever after. Since we are too lazy to actually find the N_0, we are content to verify the conjecture for say, $n \leq 200$, and call that verification a **semi-rigorous** proof.

All our proofs for the three-dimensional case are fully rigorous, but the analogous statements for four dimensions are only semi-rigorous. In the four-dimensional case it turns out that for all the starting points we looked at ($b_4 \leq b_3 \leq b_2 \leq b_1 \leq 10$), there are expressions as linear combination of negative shifts of the sequence $A_{0,0,0,0}(n)$ (with coefficients that are polynomials of degree 1 in n).

k-dimensions

Let

$$F(a_1,\ldots,a_k) := \frac{(a_1+\ldots+a_k)!}{a_1!\cdots a_k!} \quad .$$

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Recall that the number of lattice paths from the origin to $[a_1, \ldots, a_k]$ is $F(a_1, \ldots, a_k)$. Hence the number of such paths from $[b_1, \ldots, b_k]$ to $[a_1, \ldots, a_k]$ is $F(a_1 - b_1, \ldots, a_k - b_k)$. Introducing the shift operators A_i defined by $A_i F := subs(a_i = a_i + 1\,,\, F)$, this becomes $A_1^{-b_1} \cdots A_k^{-b_k} F(a_1, \ldots, a_k)$. Now André's famous and lovely reflection argument (generalized to many dimensions in [Z]) shows that the number of walks from b to a that never touch the hyperplanes $x_i - x_{i+1} = -1, i = 1 \ldots k-1$, let's call it $G_b(a)$, equals

$$G_b(a) = \sum_{\pi \in S_n} (-1)^{inv(\pi)} F(a - \pi(b)) \quad ,$$

where S_n is identified in a natural way with the group generated by the reflections about these hyperplanes, and $\pi(b)$ is the image of the point b.

It follows that there exists an easily computable *partial recurrence operator with constant coefficients* $\mathcal{P}_b(A_1, \ldots, A_k)$ (for each particular b) such that $G_b(a) = \mathcal{P}_b(A_1, \ldots, A_k) F(a_1, \ldots, a_k)$.

Back to 3 dimensions

So for any starting point b, there is an operator $\mathcal{P}_b(A_1, A_2, A_3)$ such that $G_b(a) = \mathcal{P}_b(A_1, A_2, A_3) F(a_1, a_2, a_3)$. By anti-symmetry, $\mathcal{P}_b$ is divisible by $1 - A_1 A_3^{-1}$, so we can write $\mathcal{P}_b = (1 - A_1 A_3^{-1}) \mathcal{Q}_b(A_1, A_2, A_3)$, where $\mathcal{Q}_b(A_1, A_2, A_3)$ is another (just as easily computable!) operator.

Define $H_b(a_1, a_2, a_3) = \mathcal{Q}_b(A_1, A_2, A_3) F(a_1, a_2, a_3)$, that for any *specific* b is a certain (easily-computable!) rational function of (a_1, a_2, a_3) times $(a_1 + a_2 + a_3)!/(a_1! a_2! a_3!)$. So $G_b(a_1, a_2, a_3) = H_b(a_1, a_2, a_3) - H_b(a_1 + 1, a_2, a_3 - 1)$. We now need a

Simple Lemma: Let $f(a_1, a_2, a_3)$ be any discrete function defined on $a_1 \geq a_2 \geq a_3$ that vanishes for negative arguments, and let $g(a_1, a_2, a_3) = f(a_1, a_2, a_3) - f(a_1 + 1, a_2, a_3 - 1)$, then $\sum g(a_1, a_2, a_3)$, where the sum ranges over all triples with $a_1 \geq a_2 \geq a_3 \geq 0$ and $a_1 + a_2 + a_3 = n$, equals

$$\sum_{0 \leq k < n/3} f(n - 2k, k, k) + \sum_{n/3 \leq k \leq n/2} f(k, k, n - 2k) \quad .$$

Proof: First write it as a double-sum where the outer sum is w.r.t. a_2 (ranging from 0 to $n/2$), and the inner sum w.r.t. all pairs a_1, a_3 such that $a_1 + a_3 = n - a_2$ and $a_1 \geq a_2$ and $a_3 \leq a_2$. The inner sum telescopes to $f(n - 2a_2, a_2, a_2)$ for $a_2 < n/3$ and to $f(a_2, a_2, n - 2a_2)$ for $n/3 \leq a_2 \leq n/2$.

It follows that for *any* $b = [b_1, b_2, b_3]$, $A_b(n)$ can be written as a **single-sum** $\sum_{k=0}^{n/3} H_b(n - 2k, k, k) + \sum_{k=n/3}^{n/2} H_b(k, k, n - 2k)$. For $b = [0, 0, 0]$ it turns out that (both!) summands are $F(n, k) := n!/(k!(k+1)!(n-2k)!)$, and we get Regev's famous result. For $b = [2, 1, 0]$ we get that (both!) summands, let's call it, $G(n, k)$ equals a certain (easily computable) rational function times $F(n, k)$. We have to prove that $a(n) := \sum_k G(n, k) - (F(n-1, k) - F(n-3, k))$ is identically 0, but the summand of $a(n)$, that is yet another rational function of (n, k) times $F(n, k)$, turns out to be *gosperable*, (telescoping) as Maple testifies (applying `sum` to it gives a closed-form answer), and hence $a(n)$ is indeed zero. QED.

You don't have to be a Regev to make such conjectures (once you know, thanks to Regev, where to look!), and the Maple package AMITAI, accompanying this paper, *automatically*(!) makes such conjectures for *any* starting point b, and then proceeds to *automatically*(!!) prove them, using the approach outlined above for the original Regev conjecture.

Analogous conjectures (generalizing [G], where the starting point is the origin), alas, this time with only semi-rigorous proofs, can be made for the four-dimensional case and this is also implemented in AMITAI.

Finally, we can easily use the above to derive explicit expressions (as linear combinations with constant coefficients of shifts, $M(n-i)$, of the Motzkin sequence) for the number of n-celled Standard Young tableaux with at most three rows, where the entry at *any* given cell happens to be *any* given integer. This follows from the fact that this quantity can be written as a linear combination (that AMITAI can easily figure out for itself) of the quantities $A_b(n)$ discussed previously.

The Maple package AMITAI, as well as sample input and output can be found in the webpage of this article:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/regev.html` .

In particular the file

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oAmitaiSipur310`

lists *all* the explicit expressions, in terms of the Motzkin number sequence $M(n)$, for the number of walks starting at $[b_1, b_2, 0]$ for all $0 \leq b_2 \leq b_1 \leq 10$ (all rigorously proved!), while the file

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oAmitaiSipur45`

lists *all* the explicit expressions, in terms of the sequence $A_{0,0,0,0}(n)$ (that was proved in [G] to be $C^2_{(n+1)/2}$ for n odd and $C_{n/2}C_{n/2+1}$ for n even), for the number of walks starting at $[b_1, b_2, b_3, 0]$ for all $0 \leq b_3 \leq b_2 \leq b_1 \leq 5$ (semi-rigorously proved).

Finally the file

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oAmitaiSipur3with15`

lists explicit expressions, in terms of the Motzkin numbers, for the number of three-rowed Standard Young Tableaux whose (i,j)-cell is occupied by the integer m for $1 \leq m \leq 15$ and all feasible cells (i,j) (of course $ij \leq m$).

Final Remarks

Regev[R2] raises the question of finding a *bijective* proof of his $M(n-1) - M(n-3)$ conjecture. Similar questions can be raised for each of the many, more complicated, analogs, outputted by AMITAI. It would be *interesting* to find a *uniform* way of constructing such bijective proofs. It would be even **more interesting** to prove that there is no "natural" bijection (in a certain natural sense of natural), and the identities are true **just because**.

References

[G] D. Gouyou-Beauchamps, *Standard Young tableaux of height 4 and 5*, Europ. J. Combin. **10** (1989) 69-82.

[R1] A. Regev, *Asymptotic values for degrees associated with stripes of Young diagrams*, Adv. Math. **41** (1981) 115-136.

[R2] A. Regev, *Probabilities in the (k,l) Hook*, preprint.

[S] R. Stanley, "*Enumerative Combinatorics, Volume 2*", Cambridge studies in applied mathematics **62** , Cambridge University Press (1999).

[Z] D. Zeilberger, *André's reflection proof generalized to the many-candidate ballot problem*, Discrete Math **44**, 325-326 (1983).