Proof of a Conjecture of Neil Sloane Concerning Claude Lenormand's "Raboter" Operation (OEIS sequence A318921)

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Dedicated to Neil James Alexander Sloane (born Oct. X, MCMXXXIX) on his turning 1001111 *years young (*100101 *days late)*

Yesterday, Neil Sloan gave a great talk([S1]), where among many other fascinating results and open problems, he mentioned a conjecture that he made less than two months ago, concerning the "planing" operation (*raboter*) introduced in 2004 by sequence enthusiast Claude Lenormand, and described in [S2]. He also mentioned that he has no idea how hard it is to prove, and made a 'meta-conjecture' that it may be a 'low-hanging fruit'. I will now show that both conjecture and meta-conjecture are true.

By examining the definition of `http://oeis.org/A318921` in terms of the binary representation of n, it is readily seen that $r(n)$ may be defined recursively as follows

$$r(n) \, := \, \begin{cases} 2\,r(n/2)\,, \quad if \quad n \equiv 0\,(\,mod\,4\,)\,; \\ r(\frac{n-1}{2})\,, \quad if \quad n \equiv \, 1\,(\,mod\,4\,)\,; \\ r(n/2)\,, \quad if \quad n \equiv \, 2\,(\,mod\,4\,)\,; \\ 2\,r(\frac{n-1}{2}) \, + \, 1\,, \quad if \quad n \equiv \, 3\,(\,mod\,4\,)\,, \end{cases}$$

subject to the initial conditions $r(0) = 0$ and $r(1) = 0$.

Sloane defined, for, integers k, $k \geq 1$,

$$L(k) := \sum_{n=2^k}^{2^{k+1}-1} r(n) \quad ,$$

and conjectured the following

Fact: $L(k) \, = \, 2 \, \cdot \, 3^{k-1} \, - \, 2^{k-1}$.

Indeed, breaking up the sum into the four congruence classes modulo four, we have

$$L(k) \, = \, \sum_{n=2^k}^{2^{k+1}-1} r(n)$$

$$= \sum_{m=2^{k-2}}^{2^{k-1}-1} r(4m) \, + \, \sum_{m=2^{k-2}}^{2^{k-1}-1} r(4m+1) \, + \, \sum_{m=2^{k-2}}^{2^{k-1}-1} r(4m+2) \, + \, \sum_{m=2^{k-2}}^{2^{k-1}-1} r(4m+3)$$

$$= \sum_{m=2^{k-2}}^{2^{k-1}-1} 2\,r(2m) \,+\, \sum_{m=2^{k-2}}^{2^{k-1}-1} r(2m) \,+\, \sum_{m=2^{k-2}}^{2^{k-1}-1} r(2m+1) \,+\, \sum_{m=2^{k-2}}^{2^{k-1}-1} (\,2\,r(2m+1) \,+\, 1\,)$$

$$= \,3 \sum_{m=2^{k-2}}^{2^{k-1}-1} r(2m) \,+\, \sum_{m=2^{k-2}}^{2^{k-1}-1} r(2m+1) \,+\, 2 \sum_{m=2^{k-2}}^{2^{k-1}-1} r(2m+1) \,+\, \sum_{m=2^{k-2}}^{2^{k-1}-1} 1$$

$$= \,3 \left(\sum_{m=2^{k-2}}^{2^{k-1}-1} r(2m) \,+\, \sum_{m=2^{k-2}}^{2^{k-1}-1} r(2m+1) \right) \,+\, 2^{k-2}$$

$$= \,3 \left(\sum_{m=2^{k-1}}^{2^{k}-1} r(m) \right) \,+\, 2^{k-2} \,=\, 3\,L(k-1) \,+\, 2^{k-2} \quad .$$

Hence the sequence $L(k)$ satisfies the first-order inhomogeneous recurrence with constant coefficients

$$L(k) \,-\, 3\,L(k-1) \,=\, 2^{k-2} \quad .$$

But $R(k) := \, 2 \,\cdot\, 3^{k-1} \,-\, 2^{k-1}$ also satisfies the same recurrence, i.e.

$$R(k) \,-\, 3\,R(k-1) \,=\, 2^{k-2} \quad ,$$

(check!), and the fact follows by induction on k, since it is true for $k = 1$. $\square$

References

[S1] N.J.A. Sloane, *Coordination Sequences, Planing Numbers, and Other Recent Sequences*, talk given in Rutgers University Experimental Mathematics Seminar, Nov. 15, 2018.
part 1: `https://vimeo.com/301216222` ; part 2: `https://vimeo.com/301219515` .
slides: `http://sites.math.rutgers.edu/~my237/expmath/EMNov2018.pdf` .

[S2] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, **Sequence A318921**, `http://oeis.org/A318921` .

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