HOW TO PROVE GEORGE ANDREWS' AND DAVE ROBBINS' q-TSPP CONJECTURE

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Accompanied by Maple packages TSPP and qTSPP available from
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/qtspp.html`.

Pour Pierre Leroux, In Memoriam

Preface: Montréal, May 1985

In the historic conference *Combinatoire Énumérative* [6] wonderfully organized by Gilbert Labelle and **Pierre Leroux** there were many stimulating lectures, including a very interesting one by Pierre Leroux himself, who talked about his joint work with Xavier Viennot [7], on solving differential equations combinatorially! During the problem session of that very same *colloque*, chaired by Pierre Leroux, Richard Stanley raised some intriguing problems about the enumeration of plane partitions, that he later expanded into a fascinating article [9]. Most of these problems concerned the enumeration of *symmetry classes* of *plane partitions*, that were discussed in more detail in another article of Stanley [10]. *All* of the conjectures in the latter article have since been proved (see Dave Bressoud's modern classic [2]), *except* one, that, so far, *resisted* the efforts of the greatest minds in enumerative combinatorics. It concerns the proof of an explicit formula for the q-enumeration of *totally symmetric plane partitions*, conjectured independently by George Andrews and Dave Robbins ([10], [9] (conj. 7), [2] (conj. 13)). In this tribute to Pierre Leroux, we describe how to prove that last stronghold.

1. q-TSPP: The Last Surviving Conjecture About Plane Partitions

Recall that a *plane partition* π is an array $\pi = (\pi_{ij})$, $i, j \geq 1$, of positive integers π_{ij} with finite sum $|\pi| = \sum \pi_{ij}$, which is weakly decreasing in rows and columns. By stacking π_{ij} unit cubes on top of the ij location, one gets the 3D Ferrers diagram, that can be identified with the plane-partition, and is a left-, up-, and bottom- justified structure of unit cubes, and we can refer to the locations (i, j, k) of the individual unit cubes.

A plane partition is *totally symmetric* iff whenever (i, j, k) is occupied (i.e. $\pi_{ij} \geq k$), it follows that all its (up to 5) permutations: $\{(i, k, j), (j, i, k), (j, k, i), (k, i, j), (k, j, i)\}$ are also occupied. In 1995, John Stembridge [11] proved Ian Macdonald's conjecture that the number of totally symmetric plane partitions (TSPPs) whose $3D$ Ferrers diagram is bounded inside the cube $[0, n]^3$ is given by the nice product-formula

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}.$$

Ten years after Stembridge's completely human-generated proof, George Andrews, Peter Paule, and Carsten Schneider [1] came up with a *computer-assisted* proof, that, however required lots of human ingenuity and ad hoc tricks, in addition to a considerable amount of computer time.

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Way back in the early-to-mid eighties (ca. 1983), George Andrews and Dave Robbins independently conjectured a *q-analog* of this formula, namely that the *orbit-counting generating function* ([2], p. 200, [9], p. 289) is given by

$$\prod_{1\le i\le j\le k\le n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

In this article we will show how to prove this conjecture (modulo a finite amount of routine computer calculations that may be already feasible today [with great technical effort], but that would most likely be routinely checkable on a standard desktop in twenty years).

2. Soichi Okada's Crucial Insight

Our starting point is an elegant reduction, by Soichi Okada [8], of the q-TSPP statement, to the problem of evaluating a certain "innocent-looking" determinant. This is also listed as Conjecture 46 (p. 42) in Christian Krattenthaler's celebrated essay [5] on the art of determinant evaluation.

Let, as usual, $\delta(\alpha,\beta)$ be the Kronecker delta function ($\delta(\alpha,\beta)=1$ when $\alpha=\beta$ and $\delta(\alpha,\beta)=0$ when $\alpha\neq\beta$), and let, also as usual,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)(1-q^{b-1})\cdots(1-q)}.$$

Define the discrete function $a(i,j)$ by:

$$a(i,j) = q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix} + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}\right) + (1+q^i)\delta(i,j) - \delta(i,j+1).$$

Soichi Okada ([8], see also [5], Conj. 46) proved that the q-TSPP conjecture is true if

$$\det(a(i,j))_{1\le i,j\le n} = \prod_{1\le i\le j\le k\le n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2.$$

So in order to prove the q-TSPP conjecture, all we need is to prove Okada's conjectured determinant evaluation.

3. Certificates for Determinant Identities

In [15], an empirical (yet fully rigorous!) approach is described to (symbolically!) evaluate determinants $A(n) := \det(a(i,j))_{1\le i,j\le n}$, where $a(i,j)$ is a holonomic discrete function of i and j. Note that this is an *approach*, not a method! It is not guaranteed to always work (and probably usually doesn't!).

Let's first describe this approach in more general terms, not just within the holonomic ansatz.

Suppose that $a(i,j)$ is given "explicitly" (as it sure is here), and we want to prove for *all* $n\ge 1$ that

$$\det(a(i,j))_{1\le i,j\le n} = \mathrm{Nice}(n),$$

for some *explicit* expression $\mathrm{Nice}(n)$ (as it sure is here).

The approach is to *pull out of the hat* another "explicit" (possibly in a much broader sense of the word *explicit*) discrete function $B(n,j)$, and then check the identities

(*Soichi*) $$\sum_{j=1}^{n} B(n,j)a(i,j) = 0, \qquad (1\le i<n<\infty),$$

(*Normalization*) $$B(n,n) = 1, \qquad (1\le n<\infty).$$

If we could do that, then by *uniqueness*, it would follow that $B(n,j)$ equals the co-factor of the (n,j) entry of the $n\times n$ determinant divided by the $(n-1)\times(n-1)$ determinant (that is the co-factor of the (n,n) entry in the $n\times n$ determinant). Finally one has to check the identity

$$\text{(Okada)}\qquad \sum_{j=1}^{n} B(n,j)a(n,j)=\mathrm{Nice}(n)/\mathrm{Nice}(n-1)\quad (1\le n<\infty)$$

If the suggested function $B(n,j)$ does satisfy (*Soichi*), (*Normalization*), and (*Okada*), then the determinant identity follows as a consequence. So in a sense, the explicit description of $B(n,j)$ plays the role of a *certificate* for the determinant identity.

4. The q-Holonomic Ansatz

In what sense might $B(n,j)$ be explicit? In [15] the focus was on holonomic sequences, in the present situation we will work with q-holonomic sequences. A univariate sequence $F(n)$ is called q-holonomic if it satisfies a linear recurrence of the form

$$a_r(q,q^n)F(n+r)+a_{r-1}(q,q^n)F(n+r-1)+\cdots+a_1(q,q^n)F(n+1)+a_0(q,q^n)F(n)=0$$

where $a_0,\dots,a_r$ are certain polynomials. A key feature is that $F(n)$ is uniquely determined by such a recurrence and the initial values $F(1),\dots,F(r)$. It is therefore fair to accept recurrence plus initial values as an *explicit description* of the sequence $F(n)$.

A bivariate q-holonomic sequence $F(n,m)$ is uniquely determined by a linear recurrence of the form

$$a_r(q,q^n,q^m)F(n+r,m)+\cdots+a_1(q,q^n,q^m)F(n+1,m)+a_0(q,q^n,q^m)F(n,m)=0$$

where $a_0,\dots,a_r$ are certain rational functions and $F(1,m),\dots,F(r,m)$ are q-holonomic as univariate sequences in m. This construction can be continued to discrete functions of any number of variables.

Note that while every q-holonomic discrete function can be described as above, not every function that is described as above, with *arbitrary* polynomials a_r is necessarily holonomic (usually it isn't!). However there are efficient algorithms for deciding whether a candidate discrete function given as above is holonomic or not. One empirical way of doing this is to use the description to crank many values, and then "guess" a pure recurrence with polynomial coefficients in the other variable, m, that can be routinely proved *a posteriori*.

Just as holonomic sequences [13], q-holonomic sequences have a number of important properties. We recall the most important ones:

(1) If $F(n_1,\dots,n_d)$ and $G(n_1,\dots,n_d)$ are (q-)holonomic, then so are the sequences

$$F(n_1,\dots,n_d)+G(n_1,\dots,n_d)\quad\text{and}\quad F(n_1,\dots,n_d)G(n_1,\dots,n_d).$$

A (q-)holonomic description of these can be computed algorithmically given (q-)holonomic descriptions of F and G.

(2) If (q-)holonomic descriptions of some sequences $F(n_1,\dots,n_d)$ and $G(n_1,\dots,n_d)$ are given, then it can be decided algorithmically whether $F=G$.

(3) If $F(n_1,\dots,n_d)$ is (q-)holonomic, then

$$G(n_1,\dots,n_{d-1})=\sum_{k=-\infty}^{\infty}F(n_1,\dots,n_{d-1},k)$$

is (q-)holonomic.

A (q-)holonomic description of $G(n_1,\dots,n_{d-1})$ can be computed algorithmically given a (q-)holonomic descriptions of $F(n_1,\dots,n_d)$.

5. The Computational Challenge

Denote by $B'(n,j)$ the sequence defined by (*Soichi*) and (*Normalization*). Why can we expect that $B'(n,j)$ is q-holonomic? *A priori* there is no reason why it should be. We have to *hope.* And we can systematically *search* for a potential q-holonomic description of $B'(n,j)$. If we find something, we have won, if not, we have lost, but there is always the hope that a further search, with larger parameters, would be successful.

One can use (*Soichi*) and (*Normalization*) to compute the values $B'(n,j)$ for, say, $1 \leq j \leq n \leq 35$ and then make an *ansatz* for a linear recurrence, say, of the form

$$\sum_{\gamma=0}^{10}\Bigl(\sum_{\beta=0}^{7}\sum_{\alpha=0}^{4} c_{\alpha,\beta,\gamma} q^{\alpha n} q^{\beta j}\Bigr) B'(n, j+\gamma) = 0.$$

For each specific choice of n and j, this equation reduces to a linear equation for the undetermined coefficients $c_{\alpha,\beta,\gamma}$. With different choices of j and n, we create an overdetermined linear system for the $c_{\alpha,\beta,\gamma}$. Solutions of that system, if there are any, are with good probability valid recurrences for $B'(n,j)$.

So *in principle,* we just have to solve a linear system. But *in practice,* this is not as easy as it might seem. The values of $B'(n,j)$ are rational functions in q, and so are the solutions $c_{\alpha,\beta,\gamma}$. A dense linear system over the rational functions with 440 unknowns cannot be solved directly with Gaussian elimination. The intermediate expression swell would quickly blow up the matrix coefficients to an astronomic size. Also the computation of 465 values of $B'(n,j)$ via (*Soichi*) and (*Normalization*) is not entirely for free, because it too requires solving dense linear systems whose coefficients are rational functions in q.

We solved the system using homomorphic images. In a first step, we computed the values of $B'(n,j)$ with q set to 2, and reduced modulo the prime $p := 2^{31} - 1$. This can be done quickly. Also the linear system for the ansatz above can be solved quickly within the finite field with p elements. It turned out that there is a one dimensional solution space. At this point, there is good evidence that the $B'(n,j)$ satisfy a recurrence of the above form, but we do not know the explicit form of the coefficients $c_{\alpha,\beta,\gamma}$ yet. Only their homomorphic images are known.

In the homomorphic image, 110 of the 440 coefficients $c_{\alpha,\beta,\gamma}$ are zero. We next refined the ansatz for the recurrence by discarding the terms $q^{\alpha n} q^{\beta j} B'(n, j+\gamma)$ for which $c_{\alpha,\beta,\gamma}$ was found to be zero in the homomorphic image. Next we repeated the computation of the $B'(n,j)$ and the solution of the linear system for $q = 3, 4, 5, 6, \ldots, 150$, always computing modulo p. The modular images were then combined via polynomial interpolation, rational function reconstruction, and rational number reconstruction [4] to coefficients which are rational functions in q over the integers.

The resulting candidate recurrence has a number of remarkable features.

(1) The recurrence was obtained as a solution of a dense overdetermined linear system. An artefact solution to an overdetermined system appears only with very low probability.
(2) The integer coefficients in the rational functions $c_{\alpha,\beta,\gamma}$ do not exceed 43 in absolute value. For an artefact solution, integers with absolute value up to $\sqrt{p} \approx 10^9$ would be expected with very high probability.
(3) The polynomials

$$\sum_{\beta=0}^{7}\sum_{\alpha=0}^{4} c_{\alpha,\beta,\gamma} q^{\alpha n} q^{\beta j} \qquad (\gamma = 0, \ldots, 10)$$

factorize into low degree factors. For example, the leading coefficient of the recurrence ($\gamma = 10$) factors as

$$(q^{j+6} - 1)(q^{j+10} + 1)(q^n - q^{j+9})(q^n - q^{j+10})(q^{j+n+9} - 1)(q^{j+n+10} - 1).$$

For an artefact solution, it would be expected with very high probability that all the polynomials are irreducible.

(4) The recurrence produces the correct terms of $B'(n,j)$ for values $35 < n \leq 200$ for $q = 2$ and modulo p, although these terms were not used in the computation of the recurrence. For an artefact solution, this is expected to happen with very low probability only.
(5) The recurrence produces the correct terms of $B'(n,j)$ for small n and j if q is left symbolic or set to a numeric value different from $2, 3, \ldots, 100$, For an artefact solution, this is expected to happen with very low probability only.

We have not the slightest doubt that the recurrence we found is correct. For a rigorous proof, we can define ("pull out of the head") a sequence $B(n,j)$ by a q-holonomic description consisting of the recurrence we discovered and some suitable univariate recurrences and initial values that are easy to obtain. It is contained in the Maple package `qTSPP` accompanying this article. The much easier $q = 1$ case (that would give a new proof to the already proved Stembridge theorem) is contained in the Maple package `TSPP`. Proving that $B(n,j) = B'(n,j)$ amounts to proving that $B(n,j)$ satisfies (*Soichi*). (Equation (*Normalization*) is automatically satisfied.) Thanks to algorithms of Chyzak, Salvy, Takayama [3, 12], proving (*Soichi*) *in principle* reduces to finitely many routine calculations.

Finally, (*Okada*) is also of the form $A = B$ where both sides are q-holonomic. The left side is q-holonomic because of the closure under multiplication and definite-summation, and the right side, $\mathrm{Nice}(n)/\mathrm{Nice}(n-1)$ is not just q-holonomic (a solution of *some* linear recurrence with polynomial coefficients (in q, q^n)) but in fact *closed-form* (the defining recurrence is first-order).

Summarizing, we have found a certificate $B(n,j)$ for Okada's conjectured determinant identity

$$\det(a(i,j)) = \prod_{1 \leq i \leq j \leq k \leq n} \left(\frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \right)^2 .$$

It only remains to prove rigorously that our certificate really is a certificate. While such a proof can *in principle* be carried out automatically, we found that *in practice,* i.e., with the currently available algorithms, software, and hardware, it remains a computational challenge. Even for the case $q = 1$, we are at present unable to complete the necessary non-commutative Gröbner basis computations.

6. The Semi-Rigorous Shortcut

We believe that even today, performing the computations for a rigorous proof is feasible, but it would require a huge technical effort. But why bother? First, if we wait for twenty more years, the availability of better algorithms, better software, and better hardware will probably enable us to finish up these finitely many routine calculations with no sweat. Besides, since now we know for sure that a *fully rigorous* proof exists, do we really want to see it? It won't give us any new *insight*. The beauty of the present approach is in the *meta-insight*, reducing the statement of the conjecture to a *finite* calculation. Furthermore, we know *a priori* that there exists an operator $P(q, q^n, N)$ (where N is the shift operator in n: $Nf(n) := f(n+1)$) that annihilates the difference of the left and right sides of (*Okada*). If that operator has order L, say, then a completely rigorous proof would be to check (*Okada*) for $1 \leq n \leq L$. At present, we are unable to find P, and hence do not know the value of L. But it is very reasonable that L would be less than, say, 400, and checking the first 400 cases of (*Okada*) (and analogously for (*Soichi*)) is certainly doable (we did it for $L = 100$, and $L = 400$ for TSPP, but you are welcome to go further). These are done in procedures `CheckqTSPP` in the Maple package `qTSPP`, and `CheckTSPP` in the Maple package `TSPP`, respectively. The corresponding input and output files can be found in the webpage of this article mentioned above. As a technical aside, let's confess that Maple running on our computer was only able to check (*Soichi*) and (*Okada*) for $L \leq 30$, for *symbolic* q, but for random numerical choices of q it went up to $L = 100$, and it is easy to see that with sufficiently many choices of numerical q for a given L, one can prove it for symbolic q.

In 1993, Zeilberger [14] proposed the notion of *semi-rigorous* proof. At the time he didn't have any *natural* examples. The present determinant evaluation, that was shown by Okada to imply a long-standing open problem in enumerative combinatorics, is an *excellent* example of a semi-rigorous proof that is (at least) as good as a rigorous proof. Let us conclude by promising that if any one is willing to pay us $\$10^7$ (ten million US dollars), we will be more than glad to fill in the details.

7. Postscript

The first version of this article was entitled: *"Proof of George Andrews' and Dave Robbins' q-TSPP Conjecture (modulo a finite amount of routine calculations)"*, but it was too *avant-garde* for the taste of the editors and the referees, who objected that we don't have a *rigorous proof* that we have a rigorous proof. We believe that this puritanism is outdated, but since we really want this article to be included in the special Leroux volume, we compromised.

But for those impatient people who can't wait twenty years, or cannot afford ten million US dollars, let us conclude with a sketch on how to hopefully make the present approach yield a fully rigorous proof with today's software and hardware.

The key is to take advantage of the special structure of the entries $a(i,j)$, and not just consider them as yet-another-holonomic sequence. For the sake of simplicity, let's consider the $q = 1$ case. Analogous considerations apply to the q-case.

When $q = 1$, the matrix enrty is:

$$a(i,j) = \binom{i+j-2}{i-1} + \binom{i+j-1}{i} + 2\delta(i,j) - \delta(i,j+1).$$

Let's write it as:

$$a(i,j) = a'(i,j) + 2\delta(i,j) - \delta(i,j+1) \quad ,$$

where

$$a'(i,j) = \binom{i+j-2}{i-1} + \binom{i+j-1}{i} \quad .$$

It is also helpful to define $C(n,j) := B(n, n-j)$. Note that $C(n,j)$ is defined to be 0 for $j < 0$, $C(n,0) = 1$ and $C(n,1)$ has a certain conjectured holonomic description as a sequence in n.

At this point it may be fruitful to introduce the sequence of polynomials

$$f_n(x) = \sum_{j=0}^{n} C(n,j)x^j \quad .$$

(it may be more efficient to let the sum range from $j = 0$ to $j = n+2$). The j-free recurrence for $B(n,j)$ given in the package `TSPP` translates to a certain linear recurrence equation, in n, with polynomial coefficients in (n,x), for $f_n(x)$, and the original N-free recurrence, translates to a certain *linear differential equation*, in x, with polynomial coefficients in n and x. Just like all the classical orthogonal polynomials (Legendre, Lagueree,Hermite, Jacoby, etc.), except that the relevant equations are no longer second-order, (and the $\{f_n(x)\}$ are not orthogonal). In particular the discrete-continuous function $(n,x) \to f_n(x)$ has a full holonomic description in its arguments.

Now both (*Soichi*) and (*Okada*) can be easily transcribed to certain simply-stated constant-term identities in (n,x). It is very possible, that going via this present continuous-discrete route would make the problem tractable with today's software and hardware, and we leave it as a challenge in case any of our readers is interested enough to convert our semi-rigorous proof into a fully rigorous proof, rather than wait patiently for twenty years.

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