A PROOF OF GEORGE ANDREWS' AND DAVID ROBBINS' q-TSPP CONJECTURE

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Abstract. The conjecture that the orbit-counting generating function for totally symmetric plane partitions can be written as an explicit product-formula, has been stated independently by George Andrews and David Robbins around 1983. We present a proof of this long-standing conjecture.

1. Proemium

In the historical conference *Combinatoire Énumerative* that took place at the end of May 1985, in Montreal, Richard Stanley raised some intriguing problems about the enumeration of plane partitions (see below), which he later expanded into a fascinating article [9]. Most of these problems concerned the enumeration of "symmetry classes" of plane partitions that were discussed in more detail in another article of Stanley [10]. All of the conjectures in the latter article have since been proved (see David Bressoud's modern classic [3]), except one, which until now resisted the efforts of the greatest minds in enumerative combinatorics. It concerns the proof of an explicit formula for the q-enumeration of totally symmetric plane partitions, conjectured, ca. 1983, independently by George Andrews and David Robbins ([10], [9] conj. 7, [3] conj. 13, and already alluded to in [1]). In the present article we finally turn this conjecture into a theorem.

A plane partition π is an array $\pi = (\pi_{i,j})_{1\leq i,j}$, of positive integers $\pi_{i,j}$ with finite sum $|\pi| = \sum \pi_{i,j}$, which is weakly decreasing in rows and columns so that $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$. A plane partition π is identified with its 3D Ferrers diagram which is obtained by stacking $\pi_{i,j}$ unit cubes on top of the location (i,j). This gives a left-, back-, and bottom-justified structure in which we can refer to the locations (i,j,k) of the individual unit cubes. If the diagram is invariant under the action of the symmetric group S_3 then π is called a totally symmetric plane partition (TSPP). In other words, π is called totally symmetric if whenever a location (i,j,k) in the diagram is occupied then all its up to 5 permutations $\{(i,k,j),(j,i,k),(j,k,i),(k,i,j),(k,j,i)\}$ are occupied as well. Such a set of cubes, i.e., all cubes to which a certain cube can be moved via S_3 is called an orbit; the set of all orbits of π forms a partition of its diagram (see Figure 1).

In 1995, John Stembridge [11] proved Ian Macdonald's conjecture that the number of totally symmetric plane partitions with largest part at most n, i.e., those whose 3D Ferrers diagram is contained in the cube $[0,n]^3$, is given by the elegant product-formula

$$\prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}.$$

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Figure 1. A totally symmetric plane partition with largest part 7. The yellow cubes form an orbit of length 6 (corresponding to all permutations of $(2,4,6)$), the red cubes an orbit of length 3 (all permutations of $(1,1,7)$), and the green cube on location $(4,4,4)$ represents an orbit of length 1.

Ten years after Stembridge's completely human-generated proof, Andrews, Peter Paule and Carsten Schneider [2] came up with a computer-assisted proof based on an ingenious matrix decomposition, but since no q-analog of their decomposition was found, their proof could not be extended to a proof for the q-case. A third proof of Stembridge's theorem [7], completely computer-generated, was recently found in the context of our investigations of the q-TSPP conjecture. We have now succeeded in completing all the required computations for an analogous proof of the q-TSPP conjecture, and can therefore announce:

Theorem 1. *Let π/S_3 denote the set of orbits of a plane partition π under the action of the symmetric group S_3. Then the orbit-counting generating function (*[3, p. 200], [10, p. 106]*) is given by*

$$\sum_{\pi\in T(n)} q^{|\pi/S_3|} = \prod_{1\le i\le j\le k\le n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

where $T(n)$ denotes the set of totally symmetric plane partitions with largest part at most n.

Proof sketch. Our proof is based on a result by Soichi Okada [8] who has shown that the theorem is implied by a certain—conjectured—determinant evaluation. These preliminaries are stated accurately in Section 2 below, followed by a description of the holonomic ansatz [14] that we are going to pursue. This approach relies on a kind of oracle that tells us a description of a certificate function $c_{n,j}$; the odyssey how this function has been "guessed" is described in Section 3. Once $c_{n,j}$ is known, the determinant evaluation reduces further to proving the three identities (1), (2), and (3) stated below. In Section 4 we come full circle by proving these identities. Additional technical details of the proof and our computations as well as the explicit certificates are provided electronically on our website (http://www.risc.uni-linz.ac.at/people/ckoutsch/qtspp/). □

Our proof is noteworthy not only for its obvious significance in enumerative combinatorics, as it settles a long-standing conjecture, attempted by many people. It is noteworthy also for computational reasons, as the computations we performed went far beyond what has been thought to be possible with currently known algebraic algorithms, software packages, and computer hardware.

2. The Telemachiad

In order to prove the q-TSPP conjecture, we exploit an elegant reduction by Okada [8] to the problem of evaluating a certain determinant. This determinant is also listed as Conjecture 46 in Christian Krattenthaler's essay [6] on the art of determinant evaluation.

Let, as usual, $\delta_{i,j}$ be the Kronecker delta function and let, also as usual,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

denote the q-binomial coefficient. Define the discrete function $a_{i,j}$ by:

$$a_{i,j} := q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix} + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}\right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}.$$

Okada's crucial insight is that Theorem 1 holds if

$$\det(a_{i,j})_{1\le i,j\le n} = \prod_{1\le i\le j\le k\le n} \left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 =: b_n \qquad (n\ge 1).$$

So for proving the q-TSPP conjecture, it is sufficient to prove this conjectured determinant evaluation. For this purpose we apply a computational approach originally proposed in [14] which is applicable to identities of the form

$$\det(a_{i,j})_{1\le i,j\le n} = b_n \qquad (n\ge 1)$$

where $a_{i,j}$ and b_n (with $b_n \neq 0$ for all $n\ge 1$) are given explicitly (as it is the case here).

The approach rests on the following induction argument on n. For $n=1$, the identity is trivial. Suppose the identity holds for $n-1$; then the linear system

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{n,1} \\ \vdots \\ c_{n,n-1} \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

has a unique solution $(c_{n,1}, \ldots, c_{n,n})$. The component $c_{n,j}$ of this solution is precisely the (n,j)-cofactor divided by the (n,n)-cofactor of the $n \times n$-determinant. The division is meaningful because the (n,n)-cofactor is just the $(n-1) \times (n-1)$-determinant, which by induction hypothesis is equal to b_{n-1}, which by general assumption is nonzero. Since the $n \times n$-determinant can be expressed in terms of the matrix entries $a_{n,j}$ and the normalized cofactors $c_{n,j}$ via

$$b_{n-1} \sum_{j=1}^{n} c_{n,j} a_{n,j},$$

the induction step is completed by showing that this sum evaluates to b_n.

The difficulty is that this last summation involves the function $c_{n,j}$ (of the discrete variables n and j) for which we do not have an explicit expression for general n and j. In order to achieve our goal, we guess a suitable description of a function $c_{n,j}$ and then prove that it satisfies the three identities

$$c_{n,n} = 1 \qquad (n \geq 1), \tag{1}$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0 \qquad (1 \leq i < n), \tag{2}$$

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}} \qquad (n \geq 1). \tag{3}$$

Once this is done, then by the argument given before the $c_{n,j}$ must be precisely the normalized (n,j)-cofactors of the $n \times n$-determinant and the determinant evaluation follows as a consequence. So in a sense, the function $c_{n,j}$ plays the rôle of a certificate for the determinant identity.

3. The Odyssey

In our the setting the certificate function $c_{n,j}$ will be described implicitly by a system of linear recurrence equations in n and j with coefficients depending polynomially on q, q^j and q^n. Such recurrence equations can be phrased as the elements of some noncommutative operator algebras $\mathbb{Q}(q, q^n, q^j)[S_n, S_j]$ where the symbols S_x represent the shifts $x \mapsto x+1$. If a function is annihilated by certain operators (viz. it satisfies certain recurrence equations), then it is also annihilated by all the elements in the (left) ideal generated by those operators. We speak of an annihilating ideal and represent such ideals by (left) Gröbner bases, so that for instance ideal membership can be decided effectively.

The annihilating ideals we use for representing functions are such that they uniquely determine the function up to some finitely many initial values. Technically, this means that the ideals have dimension zero and that some particular polynomial coefficients appearing in the recurrence system must not vanish simultaneously for infinitely many points (n,j). These recurrence systems are similar but somewhat simpler than q-holonomic systems [13], which satisfy some additional requirements that are not needed for our proof. We have checked that all the ideals arising in our proof are indeed of the desired form, but in the interest of clarity we suppress a more detailed description of these checks here. For a complete analysis including all technical details we refer to the supplementary material on our website. Also in the interest of clarity, we will from now on identify recurrence equations with their corresponding operators.

A priori, there is no reason why the normalized cofactors $c_{n,j}$ should admit a recursive description of the kind we are aiming at, but there is also no reason why they should not. It turns out that they do, and this is fortunate because for functions described in this way, techniques are known by which the required identities (1), (2) and (3) can be proven algorithmically [13, 12, 5, 4]. In order to find a recursive description for $c_{n,j}$, we first computed explicitly the normalized cofactors $c_{n,j} \in \mathbb{Q}(q)$ for a few hundred specific indices n and j by directly solving the linear system quoted in Section 2. Using an algorithm reminiscent of polynomial interpolation, we then constructed a set of recurrences compatible with the values of $c_{n,j}$ at the (finitely many) indices we computed.

Polynomial interpolation applied to a finite sample $u_1, \dots, u_k$ of an infinite sequence u_n will always deliver some polynomial p of degree at most $k-1$ which matches the given data. If it turns out that this polynomial matches some further sequence terms $u_{k+1}, u_{k+2}, \dots$, then it is tempting to conjecture that $u_n = p(n)$ for all n. The more specific points n are found to match, the higher is the evidence in favor of this conjecture, even though without further knowledge no finite amount of points will ever constitute a rigorous proof.

Very much analogously, it is possible to extract recurrence equations from some finite number of values $c_{n,j}$. The equations become trustworthy if they also hold for points (n,j) which were not used for their construction. In this way, we have discovered a system of potential recurrence equations for the $c_{n,j}$, which, despite being respectable in size (about 30 Megabytes), appears to be a rather plausible candidate for a recursive description of the normalized cofactors $c_{n,j}$. The system is available for download on our website.

4. The Nostos

Now we switch our point of view. We discard the definition that $c_{n,j}$ be the normalized (n,j)-cofactor and redefine the discrete function $c_{n,j}$ as the unique solution of the guessed recurrence system whose (finitely many) initial values agree with the normalized (n,j)-cofactor. If we succeed in proving that $c_{n,j}$ defined in this way satisfies the identities (1), (2) and (3), then we are done. For this purpose we provide operators which belong to the annihilating ideal of $c_{n,j}$ or related ideals and have certain features which imply the desired identities. So in a sense, these operators play the rôle of certificates for the identities under consideration.

Because of their astronomical size (up to 7 Gigabytes; equivalent to more than one million printed pages; corresponding to about 2.5 tons of paper), these certificates are not included explicitly in this article but provided only electronically on our website. Also because of their size, it was not possible to construct them by simply applying the standard algorithms from [13, 12, 4, 5]. A detailed explanation of how exactly we found the certificates is beyond the scope of this article and will be given in a separate publication. But this lack of explanation does not at all affect the soundness of our proof, because the correctness of the certificates can be checked independently by simply performing ideal membership tests. Our certificates are so big that even this "simple" calculation is not quite trivial, but a reader with a sufficient amount of patience and programming expertise will be able to do it.

We proceed by explaining the properties of the certificates provided on our website and why they imply (1), (2) and (3).

4.1. A certificate for (1).

To certify that $c_{n,n} = 1$ for all $n \geq 1$, we provide a recurrence in the annihilating ideal of $c_{n,j}$ which is of the special form

$$p_7(q, q^j, q^n)c_{n+7,j+7} + \cdots + p_1(q, q^j, q^n)c_{n+1,j+1} + p_0(q, q^j, q^n) = 0.$$

By virtue of the substitution $j \mapsto n$, it translates into a recurrence for the diagonal sequence $c_{n,n}$. This recurrence contains the annihilating operator $S_n - 1$ of the constant sequence 1 as a (right) factor, and therefore, after checking $c_{1,1} = c_{2,2} = \cdots = c_{7,7} = 1$, it can be concluded that $c_{n,n} = 1$ for all n.

Similarly, we showed that $c_{n,0} = 0$ for all $n \geq 1$ and that $c_{n,j} = 0$ for all $j > n$. This knowledge greatly simplifies the following proofs of the summation identities.

4.2. **Certificates for** (2). In order to prove the first summation identity, we translate (2) into the equivalent formulation

$$(2') \qquad \sum_{j=1}^{n} \frac{q^{i+j-1}(q^{i+j}+q^i-q-1)}{q^i-1} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix} c_{n,j} = c_{n,i-1} - (q^i+1)c_{n,i} \quad (1 \le i < n),$$

taking into account that $c_{n,0}=0$. We provide certificates for (2′).

To this end, let $c'_{n,i,j}$ be the summand of the sum on the left-hand side. A recursive description for $c'_{n,i,j}$ can be computed directly from the defining equations of $c_{n,j}$ and the fact that the rest of the summand is a q-hypergeometric factor. In the corresponding operator ideal I' we were able to find two different recurrence equations for $c'_{n,i,j}$ which are of the special form

$$p_{4,3}c'_{n+4,i+3,j} + \cdots + p_{1,0}c'_{n+1,i,j} + p_{0,1}c'_{n,i+1,j} + p_{0,0}c'_{n,i,j} = t_{n,i,j+1} - t_{n,i,j}$$

where the $p_{4,3},\ldots,p_{0,0}$ are rational functions in $\mathbb{Q}(q,q^i,q^n)$ and $t_{n,i,j}$ is a $\mathbb{Q}(q,q^i,q^j,q^n)$-linear combination of certain shifts of $c'_{n,i,j}$ (which are determined by the Gröbner basis of I' as described in [4]). Next observe that $c'_{n,i,j}=0$ for $j \le 0$ (because of the q-binomial coefficient) and also for $j>n$ (because $c_{n,j}=0$ for $j>n$). When summing the two recurrence equations for j from $-\infty$ to $+\infty$, the right-hand side telescopes to 0 and the left-hand side turns into a recurrence for the sum in (2′). This method is known as creative telescoping. Finally we obtain two annihilating operators P_1 and P_2 for the left-hand side of (2′).

For the right-hand side of (2′), we can again construct an ideal of recurrences from the defining equations of $c_{n,j}$. It turns out that this ideal contains P_1 and P_2, so that both sides of (2′) are annihilated by these two operators. Additionally, P_1 and P_2 have been constructed such that they require only finitely many initial values to produce a uniquely determined bivariate sequence. Therefore it suffices to check that the two sides of (2′) agree at some finitely many points (n,j).

4.3. **The certificate for** (3). To certify the final identity, we rewrite (3) equivalently into

$$(3') \quad (1+q^n) - c_{n,n-1} + \sum_{j=1}^{n} \frac{q^{n+j-1}(q^{n+j}+q^n-q-1)}{q^n-1} \begin{bmatrix} n+j-2 \\ n-1 \end{bmatrix} c_{n,j} = \frac{(q^{2n};q)_n^2}{(q^n;q^2)_n^2} \qquad (n \ge 1)$$

where $(a;q)_n := (1-a)(1-qa)\cdots(1-q^{n-1}a)$ denotes the q-Pochhammer symbol. As before, we use creative telescoping to provide a certified operator P which annihilates the sum on the left-hand side. In the present case, a single operator is sufficient because the sum depends only on a single variable n (there is no i there). Using the operator P and the defining equations for $c_{n,j}$, we then construct a recurrence for the entire left-hand side, which turns out to have order twelve. As this recurrence is a left multiple of the second order operator annihilating the right-hand side, the proof of Theorem 1 is completed by checking (3′) for $n=1,2,\ldots,12$. *Quod Erat Demonstrandum*!

5. Epilogue

Paul Erdős famously (and naively) believed that every short and elegant mathematical statement has a short and elegant "proof from the book", and if humans tried hard enough, they would eventually find it. Kurt Gödel, on the other hand, meta-proved that there exist many short statements whose shortest possible proof is very long. It is very possible that the q-TSPP theorem *does* have a yet-to-be-found *proof from the book*, but it is just as possible that it does not, and while we are sure that the present proof is not the shortest-possible (there were lots of random choices in designing the proof), it may well be the case that the shortest-possible proof is still very long, and would still require heavy-duty computer calculations.

Be that as it may, we believe that our general *approach*, and our way of *taming* the computer to prove something that seemed intractable with today's hardware and software are very elegant, and deserve to be included in the *book*.

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