A Sharp Upper Bound for the Order of The Recurrence Outputted by Zeilberger's Algorithm [1]

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Notation: q is an indeterminate (that commutes with everything). In this paper only, $[a]!$ is short for $(1-q)\dots(1-q^a)$, and $[a]_k$ is short for $(1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1})$. The *ground field* is $Q(q)$. The dependence on q is always understood.

q-Theorem. Let

$$F(n,k) = POL(q^n,q^k)\cdot H(n,k) \quad , \qquad (qProperHypergeometric)$$

where $POL(q^n,q^k)$ is a Laurent polynomial in (q^n,q^k), and

$$H(n,k) = \frac{\prod_{j=1}^{A}[a'_jn+a_jk+a''_j]!\ \prod_{j=1}^{B}[b'_jn-b_jk+b''_j]!}{\prod_{j=1}^{C}[c'_jn+c_jk+c''_j]!\ \prod_{j=1}^{D}[d'_jn-d_jk+d''_j]!}q^{Jk(k-1)/2}z^k \quad ,$$
$$(qPureHypergeometric)$$

where the a_j,a'_j $(1\le j\le A)$, b_j,b'_j $(1\le j\le B)$, c_j,c'_j $(1\le j\le C)$, d_j,d'_j $(1\le j\le D)$ are *non-negative integers*, and z, a''_j $(1\le j\le A)$, b''_j $(1\le j\le B)$, c''_j $(1\le j\le C)$, d''_j $(1\le j\le D)$ are *indeterminates*, and J is an integer. We also assume that the factorization in $(ProperHypergeometric)$ is *maximal*, i.e. $POL(q^n,q^k)$ is as large as possible. This entails that the difference between any of the affine-linear expressions at the top and any of those at the bottom, is never a non-negative integer. Also let

$$L = \max\left(J+\sum_{j=1}^{A}{a_j}^2\ ,\ \sum_{j=1}^{C}{c_j}^2\right)+\max\left(-J+\sum_{j=1}^{D}{d_j}^2\ ,\ \sum_{j=1}^{B}{b_j}^2\right) \quad . \qquad (qZBound)$$

There exist polynomials in q^n, $e_0(q^n),\dots,e_L(q^n)$, *not all zero*, and a rational function $R(q^n,q^k)$ such that $G(n,k):=R(q^n,q^k)F(n,k)$ satisfies

$$\sum_{i=0}^{L} e_i(q^n)F(n+i,k) = G(n,k+1)-G(n,k) \quad . \qquad (qZpair)$$

Proof: Let

$$\overline{H}(n,k) = \frac{\prod_{j=1}^{A}[a'_jn+a_jk+a''_j]!\ \prod_{j=1}^{B}[b'_jn-b_jk+b''_j]!}{\prod_{j=1}^{C}[c'_jn+c_jk+c''_j+c'_jL]!\ \prod_{j=1}^{D}[d'_jn-d_jk+d''_j+d'_jL]!}q^{Jk(k-1)/2}z^k \quad ,$$

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$$f(k) = zq^{Jk} \prod_{j=1}^{A} [a'_j n + a_j k + a''_j + 1]_{a_j} \prod_{j=1}^{D} [d'_j n - d_j k + d''_j + d'_j L - d_j + 1]_{d_j} \quad ,$$

and

$$g(k) = \prod_{j=1}^{B} [b'_j n - b_j k + b''_j - b_j + 1]_{b_j} \prod_{j=1}^{C} [c'_j n + c_j k + c''_j + c'_j L + 1]_{c_j} \quad .$$

Note that $\overline{H}(n, k+1)/\overline{H}(n, k) = f(k)/g(k)$. Write

$$G(n, k) = g(k-1) X(k) \overline{H}(n, k) \quad .$$

Substituting into $(Zpair)$ and dividing both sides by $\overline{H}(n, k)$, shows that it is equivalent to

$$f(k) X(k+1) - g(k-1) X(k) - h(q^k) = 0 \quad , \qquad (qGosper)$$

where

$$h(q^k) := \sum_{i=0}^{L} e_i(n) POL(q^n q^i, q^k) \cdot \frac{H(n+i, k)}{\overline{H}(n, k)} \quad .$$

Note that $h(q^k)$ is a *Laurent polynomial* (in q^k) since

$$\frac{H(n+i,k)}{\overline{H}(n,k)} =$$

$$\prod_{j=1}^{A} [a'_j n + a_j k + a''_j + 1]_{i a'_j} \prod_{j=1}^{B} [b'_j n - b_j k + b''_j + 1]_{i b'_j} \prod_{j=1}^{C} [c'_j n + c_j k + c''_j + i c'_j + 1]_{(L-i) c'_j} \prod_{j=1}^{D} [d'_j n - d_j k + d''_j + i d'_j + 1]_{(L-i) d'_j} \quad .$$

Let

$$M_1 := -ldeg(h) - max(-ldeg(f), -ldeg(g)) \quad , \qquad M_2 := deg(h) - max(deg(f), deg(g)) \quad .$$

We claim that $(qGosper)$ can always be solved (non-trivially) with $X(k)$ a Laurent polynomial of q^k of low-degree $-M_1$ and degree M_2. Writing

$$X(k) = \sum_{i=-M_1}^{M_2} x_i(n) (q^k)^i \quad ,$$

substituting into $(qGosper)$, and setting all the coefficients to 0, yields $-ldeg(h) + deg(h) + 1$ *homogeneous* linear equations for the $M_1 + M_2 + L + 2$ unknowns $e_0(n), \ldots, e_L(n)$, and $x_{-M_1}(n), \ldots, x_{M_2}(n)$. For such a *not-all-zero* solution to exist, we need $\#\, unknowns - \#\, equations - 1 \geq 0$, i.e. $(M_1 + M_2 + L + 2) - (-ldeg(h) + deg(h) + 1) - 1 \geq 0$, i.e. $L \geq max(deg(f), deg(g)) + max(-ldeg(f), -ldeg(g))$. But

$$deg(f) = J + \sum_{j=1}^{A} {a_j}^2 \quad , \quad -ldeg(f) = -J + \sum_{j=1}^{D} d_j \quad , \quad deg(g) = \sum_{j=1}^{C} {c_j}^2 \quad , \quad -ldeg(g) = \sum_{j=1}^{B} {b_j}^2 \quad .$$

This concludes the proof *except* that we did not rule out the possibility of $e_0(n), \ldots, e_L(n)$ being all zero (all we are guaranteed, so far, is that it is not possible for *all* of $e_0(n), \ldots, e_L(n)$, *and* $x_0(n), \ldots, x_M(n)$ to be zero). But if all the $e_i(n)$'s are zeros, then $h(k)$ is zero and (*Gosper*) becomes

$$\frac{X(k+1)}{X(k)} = \frac{g(k-1)}{f(k)} \quad .$$

Since $X(k)$ is a Laurent polynomial in q^k, it means that the roots of $f(k) = 0$ differ from the roots of $g(k-1) = 0$ by *fixed* non-negative integers, which is not possible because of the maximality hypothesis about $POL(q^n, q^k)$. Note that the maximality hypothesis always holds, automatically, whenever we have the *generic* situation with the $a_j'', b_j'', c_j'', d_j''$ arbitrary *symbols*. $\square$