Linear-Time and Constant-Space Algorithms to compute Multi-Sequences that arise in Enumerative Combinatorics (and Elsewhere)

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Abstract: How many ways, **exactly**, can a Chess King, always moving forward, i.e. with the set of steps $\{[1,0],[0,1],[1,1]\}$, walk from the origin, $(0,0)$, to the lattice point $(100000,200000)$? Thanks to the amazing Apagodu-Zeilberger extension of the Almkvist-Zeilberger algorithm, adapted in this article for combinatorial applications, this 104492-digit number, viewable from `http://www.math.rutgers.edu/~zeilberg/tokhniot/oPureRecRat2.txt`, can be computed in less than 33 seconds. But not just this particular number! Many other numbers that come up in enumerative combinatorics, and elsewhere, can be computed just as efficiently.

Maple Packages

This article is accompanied by two Maple packages, `PureRec.txt`, and `PureRecRat.txt`, available from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/PureRec.txt` and

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/PureRecRat.txt` .

The web-page of this article,

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/pure.html` ,

contains input and output files, referred to in this paper.

The purpose of this short article is to be a **pointer** to the much longer Maple packages (written by DZ) and numerous output files (generated by SBE) given in the above mentioned web-page. Recall that the Almkvist-Zeilberger algorithm [AlZ] (see [D] for an engaging exposition) produces a **pure** *linear* recurrence equation with *polynomial coefficients*, satisfied by the Taylor coefficients of many kinds of generating functions. All we need is that $R'(x)/R(x)$ is a rational function of x. This includes all rational functions, their powers, $R(x)^\alpha$ (either for symbolic or numeric exponent α) and all functions of the form $e^{P(x)}$ where $P(x)$ is a polynomial, and many other functions besides. This follows from the fact, that thanks to Cauchy, it can be written as a contour integral $\frac{1}{2\pi i}\int_{|x|=1}\frac{R(x)}{x^{n+1}}\,dx$. In fact, it can do much more, it can handle any expressions of the form $\int_{|x|=1} R(x,n)\,dx$, where $R(x,n)$ satisfies the condition that both $\frac{\frac{d}{dx}R(n,x)}{R(n,x)}$ and $\frac{R(n+1,x)}{R(n,x)}$ are **rational functions** of the discrete variable n and the continuous variable x.

Since the recurrence is **finite** order (often fairly low), this enables a very fast linear-time and constant-memory computation of such sequences. If the order of the recurrence is L, one only has to 'remember' the last L values. Of course, since the numbers (usually) grow exponentially, the *bit-size* memory is linear, but still it is very good!

After dividing by the leading term, one can write the recurrence for the sequence, let's call is $a(n)$ as

$$a(n) = \sum_{i=1}^{L} f_i(n) a(n-i) \quad ,$$

where $f_i(n)$ are **rational functions** of n. There is a potential *issue* of 'dividing by zero', but if n_0 is the largest positive integer root of the denominators, one can directly compute the first n_0 values and start the recurrence there.

The amazing **Multi-variable Almkvist-Zeilberger algorithm**, designed by Moa Apagodu and Doron Zeilberger [ApZ] extends this to several variables. If $a(n_1, \dots, n_d)$ is such a multi-sequence, gotten, for example. from the Taylor coefficients of a multi-variable rational function in d variables (and many other kinds of multi-sequences) , then the algorithm finds d **pure** recurrences, one for each discrete axis:

$$a(n_1, \dots, n_d) = \sum_{i=1}^{L_j} f_i^{(j)}(n) a(n_1, \dots, n_{j-1}, n_j - i, n_{j+1}, \dots, n_d) \quad ,$$

where $f_i^{(j)}(n_1, \dots, n_d)$ are **rational functions** of $n_1, \dots, n_d$. Then to compute a specific value, $a(n_1, \dots, n_d)$, one constructs a discrete-path from the origin $[0, \dots, 0]$ to the lattice point $(a_1, \dots, a_d)$ with d segments, one for each direction. One such path does not suffice, since in order to say, climb-up in the last axis n_d, we need L_d initial values, so we need recursively to take care of them L_d 'initial conditions', each requiring its own path in $(d-1)$-discrete space, but one only needs finitely many paths, making it linear time and constant memory.

Alas, now the **division by zero** is not so easy to overcome. We chose to only use the most obvious generic paths (see the Maple code for procedure `EvalScheme` in the Maple package `https://sites.math.rutgers.edu/~zeilberg/tokhniot/PureRec.txt`), and our program returns FAIL if one encounters a division by 0. However, we believe that this can be overcome by finding other paths that stay-away from dividing by 0.

Sample Output

For five random examples of sequences that are Taylor coefficients of one-variable functions of the form $1/P(x)^{\frac{1}{3}}$, where $P(x)$ is a randomly generated polynomial, see:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oPureRec1.txt` .

For five random examples of sequences that are Taylor coefficients of one-variable functions that are exponentials of polynomials, see:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oPureRec1a.txt` .

For five random examples of bi-sequences that are Taylor coefficients of two-variable functions of the form $1/P(x_1, x_2)^{\frac{1}{3}}$, where $P(x_1, x_2)$ is a randomly generated polynomial of x_1, x_2, see:

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oPureRec2.txt .

For five random examples of bi-sequences that are Taylor coefficients of functions that are exponentials of polynomials of two variables, see

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oPureRec2a.txt .

For three random examples of triple-sequences that are Taylor coefficients of a three-variable function of the form $1/P(x_1, x_2, x_3)^{\frac{1}{3}}$, where $P(x_1, x_2, x_3)$ is a randomly generated polynomial of x_1, x_2, x_3, see:

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oPureRec3.txt .

Now it takes much longer to *generate* the scheme, but once found, it is still very fast to compute specific values, even those very far from the origin,

For three random examples of tri-sequences that are Taylor coefficients of functions that are exponentials of polynomials of three variables, see

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oPureRec3a.txt .

Here we encountered some singularities, so our (incomplete) implementation sometimes (but fairly rarely) returns FAIL.

Applications to the Enumeration of Lattice Path in the Plane with Many choices of Atomic Steps

Let `St` be an arbitrary set of positive steps, e.g. for the forward-moving Chess King it is $\{[1,0],[0,1],[1,1]\}$. How can we compute fast the number of ways of walking from the origin to $[a,b]$ using the members of `St`?

The lattice-path oriented companion Maple package

https://sites.math.rutgers.edu/~zeilberg/tokhniot/PureRecRat.txt ,

has a procedure

`Walk2Dpaper(St,K)` ,

that inputs an arbitrary set of positive steps in 2-dimensions, and a large positive integer K (for illustrative purposes) and outputs the scheme for the bi-variate function $F(a,b)$, defined as the number of walks from $[0,0]$ to $[a,b]$ using the steps in `St`, and if no singularity is encountered, outputs $F(K,2K)$. It also outputs the one-variable linear recurrence for the diagonal sequence $\{F(a,a)\}$ and uses it to compute $F(2K,2K)$ (now there are never singularities). For the sake of Neil Sloane, it also spits out the first 30 terms of the diagonal sequence.

You are welcome to try out procedure `Walk2Dpaper(St,K)` with $St = \{[0,1],[1,0],[1,1]\}$ and $K = 100000$. getting the 104922-digit number mentioned in the abstract.

For 29 such theorems see:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oPureRecRat1.txt` .

Christoph Koutschan's even better program (alas, in Mathematica)

A Mathematica implementation of the multi-variate Almkvist-Zeilberger algorithm is contained, *inter alia*, in Christoph Koutschan's very efficient and versatile Mathematica program [K]. It would be nice if he would use the simple ideas in our paper to interface them and create such a *multi-sequence calculator*.

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Exclusively published in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger and arxiv.org

March 9, 2022