Remarks On The PARRONDO PARADOX

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The celebrated *Parrondo Paradox* (Harmer and Abbott, Nature **402** (Dec. 23/30 1999), p. 864 and its references, see also Paulos's gripping account in www.ABCnews.com) loses some of its shock value if instead of the 'gambling' losing games[2] A and B, whose combinations (e.g. AABB) turned out to be winning, we take the following completely deterministic 'games'. Suppose that your current capital is n dollars. Then

Game Y (resp. Z) : If n is even (resp. odd) you WIN 1 dollar, otherwise you LOSE 3 dollars.

Only playing Y, or *only* playing Z, results in a steady loss of one dollar per move, but playing YZYZYZYZ... (starting with 0 dollars) results in a steady win of 1 dollar per move. So Parrondo's paradox seems to be saying that it is better to play $(YZ)^n$ rather than Y^nZ^n, in other words, order matters!

But this is hardly a new message. We were all taught to *think* before *acting*, to *measure* before *cutting*, to *study* before *playing* and so on. Also that *variety is the spice of life*. Also *timing is everything*, to sell a stock when it is high, and buy it when it is low, (etc.)*.

Going back to the above 'games' Y and Z, they still constitute a 'paradox' if you replace the loss of three dollars by the loss of one dollar. Now both Y and Z are *fair*, i.e. you neither win nor lose, yet playing YZ in succession (starting with 0 dollars) still guarantees a steady win. Here is a variant of Paulos's very apt spatial analog. Suppose that we have infinitely many cities labelled by the integers. The train company Y (resp. Z) has service from city n to city $n+1$ and back to n for n even (resp. odd). If you only buy Y tickets, or only buy Z tickets, you'll be going back and forth between your starting city and its immediate neighbor, but if you buy both kinds of tickets, and alternate between them, then you can go as far as you want. But most of us use this wisdom every day when we alternate between *walking*, *driving*, and *flying*.

The mathematical reason for the general Parrondo paradox is the fact that matrices (usually) do not commute. This brings to mind Quantum Mechanics with its own chock-full of 'paradoxes'.

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[2] If your current capital is n dollars then in game A you win or lose one dollar with probabilities $1/2-e$ and $1/2+e$ respectively, while in game B you win or lose one dollar with probabilities $p_1 := 1/10 - e$ and $1 - p_1 = 9/10 + e$ resp. if n is divisible by 3 otherwise you win or lose one dollar with probabilities $p_2 := 3/4 - e$ and $1 - p_2 = 1/4 + e$ resp.

The Maple Package PARRONDO Tells You the BEST Periodic Strategy

The Maple package `PARRONDO` far extends the mathematical analysis of Harmer, Abbott and Taylor (Proc. Royal Soc., to appear). With A and B of the original Parrondo paradox, and $e = 0$ (i.e. both A and B are fair), the best strategy is to play $(ABBAB)^*$ with a steady win of 7.5646 cents, about three times the amount one obtains by playing $(AABB)^*$, that only yields 2.4539 cents. This is still true when $p_2 = 3/4$ is replaced by any other $p_2 > 1/2$. The best pay-off is when game B becomes deterministic with $p_2 = 1$, resulting (still with playing $(ABBAB)^*$) with a steady gain of 36 cents per move. When $p_2 \leq 1/2$ then any non-trivial combination of A and B is losing, so in this case *mixing is bad*, and one should stick to *only* A , or *only* B.

For specific p_2, period-length n, 'modolus' M, and bias e, the function call is `GBestWord(n,A,B,M,`p_2`,e);`. For example, in the original fair case one should type `GBestWord(5,A,B,3,3/4, 0);` in order to get the best period-5 strategy. If the bias is $e = 1/100$ then one should type `GBestWord(5,A,B,3,3/4, 1/100);`. In either case Maple responds with the set consisting of `[A,B,B,A,B]` and its cyclic shifts, followed by the (asymptotic) gain per move.

Not as good as $(ABBAB)^*$, but far better than $(AABB)^*$, is the period-3 strategy $(ABB)^*$, that yields 6.786 cents. Finally picking A or B at random with probabilities p and $1 - p$ resp. is optimized when $p = .4145$ but even with that optimal value it only yields 2.62 cents (do `Bestp(0);` in `PARRONDO`).

Finally, if you replace $M = 3$ by $M = 4$ (i.e. instead of having the 'bad luck' amounts be multiples of 3, now they are multiples of 4), then `PARROND` (type e.g. `GBestWord(12,A,B,4,3/4, 0);`) tells us that the best strategy is $(AB)^*$.

We hope that other Parrondoholics will not be paranoid, and use our Maple package to find further interesting properties of this, not quite paradoxical, but yet very interesting, phenomenon called the Parrondo paradox.