A Simple Re-Derivation of Onsager's Solution of the 2D Ising Model using Experimental Mathematics

Manuel KAUERS and Doron ZEILBERGER

Abstract: In this *case study*, we illustrate the great potential of experimental mathematics and symbolic computation, by rederiving, **ab initio**, Onsager's celebrated solution of the two-dimensional Ising model in zero magnetic field. Onsager's derivation is extremely complicated, as are all the subsequent proofs. Unlike Onsager's, our derivation is not rigorous, yet it is *absolutely certain* (even if Onsager did not do it before), and should have been acceptable to physicists who do not share mathematicians' fanatical (and often misplaced) insistence on rigor.

Two Warm-Up Exercises

Definition 1: For an $n_1 \times n_2$ matrix, $M = (m_{i,j})$, and any positive real numbers x and y:

$$weight(M)\,(x, y) \;:=\; x^{\frac{1}{2}\left(\sum_{i,j} m_{i,j}\, m_{i+1,j} + m_{i,j}\, m_{i,j+1}\right)} \cdot y^{\sum_{i,j} m_{i,j}} \quad .$$

(We make the convention that if i is n_1, then $i + 1 = 1$, and if $j = n_2$ then $j + 1 = 1$.)

Definition 2: Let $\mathcal{M}(n_1, n_2)$ be the set of $n_1 \times n_2$ matrices whose entries are either 1 or -1 (of course, there are $2^{n_1 n_2}$ such matrices). The Laurent polynomial $P_{n_1,n_2}(x, y)$ is defined as follows.

$$P_{n_1,n_2}(x, y) \;:=\; \sum_{M \in \mathcal{M}(n_1,n_2)} weight(M)\,(x, y) \quad .$$

Definition 3: For x, y positive real numbers:

$$f(x, y) \;:=\; \lim_{n\to\infty} \frac{\log\, P_{n,n}(x, y)}{n^2} \quad .$$

Exercise 1: Find an explicit, **closed-form**, expression for $f(x, y)$.

Definition 1a: For an $n_1 \times n_2 \times n_3$ three-dimensional array, $M = (m_{i,j,k})$, and a positive real number x,

$$weight'(M)\,(x) \;:=\; x^{\frac{1}{2}\left(\sum_{i,j,k} m_{i,j,k}\, m_{i+1,j,k} + m_{i,j,k}\, m_{i,j+1,k} + m_{i,j,k}\, m_{i,j,k+1}\right)} \quad .$$

Definition 2a: Let $\mathcal{M}(n_1, n_2, n_3)$ be the set of $n_1 \times n_2 \times n_3$ three-dimensional arrays, whose entries are either 1 or -1 (of course, there are $2^{n_1 n_2 n_3}$ such arrays), define the Laurent polynomial in x, by

$$Q_{n_1,n_2,n_3}(x) \;:=\; \sum_{M \in \mathcal{M}(n_1,n_2,n_3)} weight'(M)\,(x) \quad .$$

Definition 3a: For x, a positive real number,

$$g(x) \; := \; \lim_{n \to \infty} \frac{\log \, Q_{n,n,n}(x)}{n^3} \quad .$$

Exercise 1a: Find an explicit, **closed-form**, expression for $g(x)$.

We hope, dear readers, that you will spend *some* time trying to solve these two exercises, but please do not spend too much time! While we do know that the limits exist [vH], evaluating them explicitly has been an open problem for almost eighty years, and in spite of many attempts by the best minds in mathematical physics, both 'excercises' are still wide open.

Exercise 1 is called "solving the two-dimensional Ising model with magnetic field", while Exercise 1a is called "solving the three-dimensional Ising model in zero magnetic field". Let us quote Ken Wilson, who got the Physics Nobel prize in 1982 for seminal (*non-rigorous*!) work on questions related to these two 'exercises'.

"When I entered graduate school I had carried out the instructions given to me by my father [notable chemist E. Bright Wilson, who co-authored, with Linus Pauling, the classic Introduction to Quantum Mechanics] *and had knocked on both Murray Gell-Mann's and Feynman's doors and asked them what they were currently doing. Murray wrote down the partition function for the three-dimensional Ising model and said that it would be nice if I could solve it. Feynman's answer was 'nothing' . "* [Quoted in Julia Yeomans' wonderful book [Y], p. 35 .]

Onsager's Solution

In 1944, Lars Onsager famously derived, and *rigorously* proved, the special case of Exercise 1, when $y = 1$.

Onsager's Explicit Formula For the Zero-Field 2D Ising Model: Let

$$G(z) \; := \; -\frac{1}{4} \sum_{r=1}^{\infty} \binom{2r}{r}^2 \frac{z^{2r}}{r} \quad ,$$

then

$$f(x,1) \; = \; \ln(x + x^{-1}) \, + \, G\left(\frac{x - x^{-1}}{(x + x^{-1})^2} \right) \quad .$$

Onsager's proof [O], and all the subsequent proofs, are very complicated. We will soon show how this formula could have been *naturally* derived, way back in 1941, if they had the software and hardware that we have today (and even, probably, thirty years ago).

Unlike Onsager's derivation, that is fully rigorous, ours is not. So from a strictly (currently mainstream) mathematical viewpoint, it would have been considered 'only' a conjecture, were it done before Onsager's rigorous derivation. But this conjecture would have been **so** plausible that it would have been whole-heartedly accepted by the theoretical physics community.

What is an "Explicit" Answer

From now on we will write $f(x)$ instead of $f(x,1)$, and $P_{n_1,n_2}(x)$ instead of $P_{n_1,n_2}(x,1)$.

Onsger's elegant solution involves an infinite series, that entails taking a limit. The *definition* of the function $f(x)$ also involves taking a limit (namely of $\frac{\log(P_{n,n}(x))}{n^2}$ as $n \to \infty$). Why is the former limit better than the latter?

Indeed, the notion of "explicit", or "closed form" is vague and cultural. In ancient Greece a geometrical construction was acceptable only if it used ruler and compass. In algebra, for a long time, a solution was acceptable only if it could be expressed in terms of the four elementary operations and root extractions. In enumerative cominatorics, a solution was (and sometimes still is) considered closed form only if it is a product and/or quotient of factorials, and there are many other examples.

In a famous *position paper* [W], Herb Wilf tackled this problem in combinatorics. He was inspired to write it when he was asked to referee a paper containing a "formula" for a certain quantity. It turned out that computing the quantity via the formula took much longer than using the definition. Inspired by the—at the time—new paradigm of "computational complexity", he suggested that an "answer" is an efficient algorithm to compute the quantity in question.

How would we compute $f(x)$, using the definition, for a specific, 'numeric', x? We can, in principle, compute the sequence of Laurent polynomials $P_{n,n}(x)$ directly, for, say, $n \leq 30$, and get the finite sequence of numbers $\{\log P_{n,n}(x)/n^2\}_{n=1}^{30}$, and see whether they get closer-and-closer, and estimate the limit. Alas, computing $P_{n,n}(x)$ by brute force involves adding up 2^{n^2} terms, each of which take $O(n^2)$ operations to compute. This is hopeless! Also, to be fully rigorous, one has to be able to find *a priori* bounds for the error, and for each ϵ find (rigorously) an n_ϵ such that $|f(x) - \log(\, P_{n,n}(x)\,)/n^2| < \epsilon$ for $n \geq n_\epsilon$. This is truly hopeless.

On the other hand, using elementary calculus, Onsager's solution enables us to compute $f(x)$, very fast, to any desired accuracy.

More importantly, physicist do not really care about the explicit form of $f(x)$ (or more generally, the still wide open $f(x,y)$, and $g(x)$), they want to know the *exact* location of the **singularities**, (critical points) that describes at what value of x (and hence at what temperature) a *phase transition* occurs, e.g. at what temperature water boils or freezes. Even more importantly, they care about the **nature** of the singularities, in other words, *how* water boils rather than at what temperature (that depends, e.g. on pressure). From Onsager's solution, one can easily find, using Calculus I, the location, and nature, of the singularity of $G(z)$, and hence of $f(x)$. It is impossible to extract this information directly from the definition.

This motivation may be interesting, but it is irrelevant to us. All we want is to answer exercise 1 in the special case $y = 1$, with as little effort as possible, and making full use of the computer. We only require elementary calculus and very elementary matrix algebra. We don't even use eigenvalues!

Recommended Reading

Even though it is *irrelevant* to our story, for those readers who **do** wish to know the context and background, we strongly recommend Barry Cipra's [C] very lucid and very engaging introduction to the Ising model. We also recommend the excellent books [T] and [Y].

Symbol-Crunching

Of course, it would be nice to find an *expression* for $f(x)$ in terms of the **symbol** x. Computing $P_{n,n}(x)$ for any specific n is a finite (albeit **huge**) computation, involving summing 2^{n^2} monomials, so we can't go very far. But, let's assume that we live in an ideal world, or that quantum computing became a reality, then computing $P_{n_1,n_2}(x)$, and in particular, $P_{n,n}(x)$, being finite, is always possible. The first, very natural, step, already proposed in 1941, that was motivated by the combinatorial approach (see later, and [T], Ch.6, Eq. 1.9, where we replace x^2 by x) is to write

$$P_{n_1,n_2}(x) = \frac{(x+2+x^{-1})^{n_1\,n_2}}{2^{n_1\,n_2}} Z_{n_1,n_2}(w) \quad , \quad \text{where} \quad w = \frac{x-1}{x+1} \quad .$$

It follows from a simple combinatorial argument that $Z_{n_1,n_2}(w)$ is a *polynomial* in w, of degree $n_1\,n_2$.

Taking logarithms, and dividing by $n_1\,n_2$, we get

$$\frac{\log P_{n_1,n_2}(x)}{n_1\,n_2} = -\log 2 + \log(x^{-1}+2+x) + \frac{\log Z_{n_1,n_2}(w)}{n_1\,n_2} \quad .$$

Using the fact (do it!) that $x^{-1}+2+x = \frac{4}{1-w^2}$ we get that

$$f(x) = \log 2 \; - \; \log(1-w^2) + \lim_{n\to\infty} \frac{\log Z_{n,n}(w)}{n^2} \quad \text{where} \quad w = \frac{x-1}{x+1}.$$

So from now, all we need is to find

$$F(w) := \lim_{n\to\infty} \frac{\log Z_{n,n}(w)}{n^2} \quad .$$

Now, it turns out (and it follows from elementary considerations) that the sequence $\frac{\log Z_{n,n}(w)}{n^2}$ converges in the sense of 'formal power series'. More precisely, for any positive integer, r, the coefficient of w^r in $F(w)$ (our object of desire) coincides with that of $\frac{\log Z_{n,n}(w)}{n^2}$ as soon as $n > r$. So a natural *experimental mathematics* approach would be to try and find as many Taylor coefficients of $F(w)$ as our computer would allow and look for a *pattern* that would enable us to conjecture a closed-form expression for the Taylor coefficients of $F(w)$, thereby determining $F(w)$ and hence $f(x)$.

In an ideal world, with an indefinitely large computer, this very naive approach would have succeeded. Alas, as it turned out, we would have needed to compute $P_{n,n}(x)$ for $n = 96$, and since

2^{96^2} is such a big number, this very naive **brute force** approach is doomed to failure in our tiny universe.

Using Transfer Matrices

A much more efficient approach to computing the Laurent polynomials $P_{n_1,n_2}(x)$ (and hence the polynomials $Z_{n_1,n_2}(w)$), was suggested in the seminal paper of Kramers and Wannier [KW]. That was also Onsager's starting point. It is easy to see (see [T], p. 118) that for each n_1, there are easily computed 2^{n_1} by 2^{n_1} matrices, let's call them $A_{n_1}(x)$ such that

$$P_{n_1,n_2}(x) = \mathrm{Trace}\, A_{n_1}(x)^{n_2} \quad .$$

With today's computers, it is possible to compute these for $n_1 \leq 12$ and as large as n_2 as desired.

But once again, one can (still) not go very far.

In 1941, B.L. van der Waerden suggested an ingenious (very elementary!) combinatorial approach, described beautifully in Barry Cipra's article [C] (see also Chapters 6 of [T] and [Y] for nice accounts). He observed that the coefficients of w in the polynomial $Z_{n_1,n_2}(w)$ have a nice combinatorial interpretation. Putting $N = n_1 n_2$, it turned out (and is very easy to see, see [T]) that for any positive integer r, the coefficient of w^r in $Z_{n_1,n_2}(w)$, let's call it p_r, is the number of 'lattice polygons' with r edges that can lie in an n_1 by n_2 'torodial rectangle', i.e. the set $\{0, \ldots, n_1\} \times \{0, \ldots, n_2\}$ with 0 identified with n_1 and n_2 respectively. A lattice polygon is a collection of edges such that every participating vertex has an *even* number (0, 2, or 4) of neighbors. It follows in particular that p_r is zero if r is odd.

It also follows from elementary combinatorial considerations that for $n_1 > r, n_2 > r$, the coefficient p_r is a certain *polynomial* in N ([T], p. 150, Eq. (1.17)), and hence may be written $p_r(N)$, and we can write:

$$p_r(N) = N a_r^{(1)} + N^2 a_r^{(2)} + \ldots + N^m a_r^{(m)} \quad .$$

Now it also follows from elementary considerations, already known in 1941, that once you take the log, divide by $N = n_1 n_2$ and take the limit, *only* the coefficients of N in these 'Ising polynomials' survive, and that

$$F(w) = \lim_{n\to\infty} \frac{\log(\, Z_{n,n}(w)\,)}{n^2} \; = \sum_{r=1}^{\infty} a_r^{(1)} w^r \quad .$$

It remains to compute as many Ising polynomials, $p_r(N)$, as our computers will allow us, extract the coefficients $a_r^{(1)}$ of N, and hope to detect a *pattern*, to enable us to conjecture the general coefficient of $F(w)$, and hence know $f(x)$.

How to compute the Combinatorial Ising Polynomials?

The first thing that comes to mind, and works well for small r is to actually look for the kind of lattice polygons that can show up, but as r gets larger, this gets out of hand. Rather than do the intricate combinatorics, we use the fact that $P_{n_1,n_2}(x) = \mathrm{Trace}\, A_{n_1}(x)^{n_2}$, from which we can compute $Z_{n_1,n_2}(w)$ for $n_1 \le 12$ (say) and n_2 as large as desired. For each individual coefficient of w^r (r even), we output it for sufficiently many specific n_1 and n_2, and then using *undetermined coefficients* or interpolation we fit them into a polynomial (whose degree we know beforehand). In fact, it is possible to get $p_{2r}(N)$ by looking at $n_1 = r-2, n_2 > r$, by excluding obvious polygons that belong to the $(r-2)\times n_2$ torodial rectangle but are impossible for a larger rectangle.

The Ising Polynomials

By using this very naive approach (only using matrix multiplication and then taking the trace) our beloved computers came up with the following first 10 Ising polynomials (we were able to find quite a few more, but as we will soon see, the first ten polynomials suffice).

$$p_2(N) = 0, \quad p_4(N) = N, \quad p_6(N) = 2\,N, \quad p_8(N) = \frac{1}{2}\,N\,(9+N)\,, \quad p_{10}(N) = 2\,N\,(6+N)\,,$$

$$p_{12}(N) = \frac{1}{6}\,N\,(7+N)\,(32+N)\,, \quad p_{14}(N) = N\left(130 + 21\,N + N^2\right),$$

$$p_{16}(N) = \frac{1}{24}\,N\left(11766 + 1715\,N + 102\,N^2 + N^3\right), \quad p_{18}(N) = \frac{1}{3}\,N\left(5876 + 776\,N + 49\,N^2 + N^3\right),$$

$$p_{20}(N) = \frac{1}{120}\,N\left(980904 + 118830\,N + 7415\,N^2 + 210\,N^3 + N^4\right) \quad .$$

Extracting the coefficients of N, we get

$$0, 1, 2, \frac{9}{2}, 12, \frac{112}{3}, 130, \frac{1961}{4}, \frac{5876}{3}, \frac{40871}{5} \quad .$$

Hence $F(w)$ starts with

$$F(w) = w^4 + 2\,w^6 + \frac{9}{2}\,w^8 + 12\,w^{10} + \frac{112}{3}\,w^{12} + 130\,w^{14} + \frac{1961}{4}\,w^{16} + \frac{5876}{3}\,w^{18} + \frac{40871}{5}\,w^{20} + \cdots \quad .$$

However, these ten terms (and even forty of them) do not suffice to guess a pattern.

Duality Saves the Day

Way back in 1941, in the seminal paper of Kramers and Wannier, that we have already mentioned, they discovered the *duality relation* (see [C] for a lucid explanation)

$$f\left(\frac{x+1}{x-1}\right) = f(x) - \log\left(\frac{x - x^{-1}}{2}\right) \quad .$$

Letting

$$x^* = \frac{x+1}{x-1} \quad ,$$

the duality relation can be written as

$$f(x^*) = f(x) - \log\left(\frac{x - x^{-1}}{2}\right) \quad ,$$

or in a more symmetric form

$$f(x) - \log(x + x^{-1}) \;=\; f(x^*) - \log(x^* + (x^*)^{-1}) \quad .$$

It follows that a more natural, and hopefully user-friendly, function to consider is

$$\bar{f}(x) := f(x) - \log(x + x^{-1}) \quad ,$$

and we have that $\bar{f}(x)$ is unchanged under the involution $x \leftrightarrow x^*$,

$$\bar{f}(x^*) = \bar{f}(x) \quad .$$

It is natural to change from the variable w to one that is invariant under the change $x \leftrightarrow x^*$. There are many possibilities. Obviously, in order to ensure the invariance, we can set $z = R(x, x^*)$ for any *symmetric* rational function R. We only need to ensure that when w is expressed as a series in z, this series has positive order, so that we are allowed to substitute it into $F(w)$. Since $F(w)$ has only even exponents, we may also prefer that the series $w = w(z)$ has only odd exponents in z, so that the substitution does not introduce odd exponents into $F(w)$.

If we try a generic *template* ('ansatz') for a symmetric rational function with numerator and denominator of degree at most two,

$$z = \frac{a_{0,0} + a_{1,0}(x + x^*) + a_{0,1}xx^* + a_{2,0}(x + x^*)^2 + a_{1,1}(x + x^*)xx^* + a_{0,2}(xx^*)^2}{b_{0,0} + b_{1,0}(x + x^*) + b_{0,1}xx^* + b_{2,0}(x + x^*)^2 + b_{1,1}(x + x^*)xx^* + b_{0,2}(xx^*)^2}$$

where $a_{i,j}$ and $b_{i,j}$ are undetermined coefficients, we get a system of polynomial equations that can be easily solved using so-called Gröbner bases. This gets translated into an equation relating z and w by eliminating x, using the fact that $x = \frac{1+w}{1-w}$. The (computer-generated) result is an equation of the form

$$(\ldots) + (\ldots)w + (\ldots)w^2 + (\ldots)w^3 + (\ldots)w^4 + (\ldots)z + (\ldots)wz + (\ldots)w^2z + (\ldots)w^3z + (\ldots)w^4z = 0 \quad ,$$

where the dots stand for certain linear combinations of the undetermined coefficients which we suppress here because of their size. In order to ensure that the solution for w of this equation is a series in z with odd exponents only, it suffices to force the coefficients of all terms $w^i z^j$ with $i + j$ even to zero. This gives a linear system whose solution brings the equation down to

$$(w - 1)w(w + 1)(a_{0,0} + a_{0,1} + a_{0,2}) + (1 + w^2)^2 z(b_{0,0} - b_{1,0} + b_{2,0}) = 0.$$

This suggests the choice

$$z \;=\; \frac{cw(1 - w^2)}{(1 + w^2)^2}, \quad \text{or} \quad w \;=\; \frac{z}{c} + \frac{3z^3}{c^3} + \frac{22z^5}{c^5} + \frac{211z^7}{c^7} + \frac{2306z^9}{c^9} + \cdots,$$

for some nonzero constant c. The value of c is not important. We take $c = 2$.

Let $\bar{f}(x)$, in terms of w, be written $\bar{F}(w)$, then (since $x+x^{-1}=\frac{2(1+w^2)}{(1-w^2)}$; note that $x=\frac{1+w}{1-w}$)

$$\bar{F}(w) := f(x) - \log(x+x^{-1}) = -\log(1-w^2) + F(w) + \log 2 - \log\left(\frac{2(1+w^2)}{1-w^2}\right)$$

$$= -\log(1+w^2) + \sum_{r=0}^{\infty} a_r^{(1)} w^r \quad ,$$

giving

$$\bar{F}(w) = -w^2 + \frac{3}{2}w^4 + \frac{5}{3}w^6 + \frac{19}{4}w^8 + \frac{59}{5}w^{10} + \frac{75}{2}w^{12}$$

$$+\frac{909}{7}w^{14} + \frac{3923}{8}w^{16} + \frac{17627}{9}w^{18} + \frac{81743}{10}w^{20} + O(w^{22}) \quad .$$

Changing the variable to z, and renaming $\bar{F}(w)$ to $G(z)$, we get

$$G(z) = -\frac{1}{4}z^2 - \frac{9}{32}z^4 - \frac{25}{48}z^6 - \frac{1225}{1024}z^8 - \frac{3969}{1280}z^{10} - \frac{17787}{2048}z^{12}$$

$$-\frac{184041}{7168}z^{14} - \frac{41409225}{524288}z^{16} - \frac{147744025}{589824}z^{18} - \frac{2133423721}{2621440}z^{20} + O(z^{22}) \quad .$$

The first ten terms of the sequence of coefficients, let's call them $\{b_{2r}\}_{r=1}^{10}$

$$-\frac{1}{4}, -\frac{9}{32}, -\frac{25}{48}, -\frac{1225}{1024}, -\frac{3969}{1280}, -\frac{17787}{2048}, -\frac{184041}{7168}, -\frac{41409225}{524288}, -\frac{147744025}{589824}, -\frac{2133423721}{2621440}, \ldots$$

factorize nicely, which indicates that the series might be **hypergeometric**, i.e. the ratio of consecutive terms is a rational function of r. This is good news, since the famous sine and cosine functions and many other functions that come up in physics (e.g. the Hermite and Legendre polynomials that are so important in Quantum Mechanics) and elsewhere are hypergeometric.

By setting up an 'ansatz'

$$\frac{b_{2r+2}}{b_{2r}} = \frac{a_0 + a_1 r + a_2 r^2 + a_3 r^3}{b_0 + b_1 r + b_2 r^2 + r^3} \quad ,$$

plugging-in the known values for $1 \le r \le 9$, and simplifying, we get a system of nine linear equations with the seven unknowns $a_0, a_1, a_2, a_3, b_0, b_1, b_2$. If you take a random such system it is most likely unsolvable. If the computer finds a solution, it is great news. What is true for the first 9 values is probably true for ever.

This means that the sequence of **ratios** b_{2r+2}/b_{2r} probably matches a rational function in r. Given the ratios $\{b_{2r+2}/b_{2r}\}_{r=1}^{9}$, the computer immediately established that

$$\frac{b_{2r+2}}{b_{2r}} = \frac{r(2r+1)^2}{(r+1)^3} \quad ,$$

for $1 \le r \le 9$, and if true for all r, this would imply the closed-form expression, for the coefficients

$$b_{2r} = -\frac{\binom{2r}{r}^2}{r4^{r+1}} \quad .$$

Since we can (nowadays!) easily extend the sequence b_{2r} up to (at least) sixteen terms, and this 'guess' indeed continued to hold, this makes it virtually certain that the guess is correct. Combining everything, we derived, *ab initio*, by *pure guessing* (and very elementary and natural reasoning), Onsager's formidable formula. ■

What's next?

Now that we have rediscovered Onsager's explicit formula for $f(x) = f(x,1)$, a natural next step towards the general case $f(x,y)$ is to determine an explicit expression for $m(x) = \frac{d}{dy} f(x,y)|_{y=1}$, i.e., the next term in the Taylor series expansion of $f(x,y)$ with respect to y at $y = 1$. Physicists call this the *"spontaneous magnetization"*.

Using transfer matrices, as before, it is easy to compute the first few terms of $m(x)$ as a series in x (or w, or z), and we don't even need a computer to guess an explicit expression for them: they all are zero. But that's just a part of the story.

Onsager observed that $m(x)$ is only zero for $x < 1 + \sqrt{2}$, while for $x \geq 1 + \sqrt{2}$, it is equal to

$$\left(\frac{(x^2+1)^2(x^2-2x-1)(x^2+2x-1)}{(x-1)^4(x+1)^4} \right)^{1/8} \quad .$$

According to Thompson ([T], p. 135), this expression *"was first derived by Onsager in the middle of the 1940s, but in true Onsager fashion he has not to this day published his derivation"*.

We don't know how he found this expression, but here is one way one could search for it, using experimental mathematics. For specific numbers x, y, we can compute numerical approximations of $f(x,y)$ using the original definition (Def. 3 above). For example, taking $f(x,y) \approx \log P_{n,n}(x,y)/n^2$ with $n \approx 20$ gives several correct digits at a reasonable computational cost. From the numerical estimates of $f(x,y)$ for various points x, y, we can obtain numerical estimates for $m(x)$ and $m'(x)$, for various points x.

The idea is to fit a differential equation against this numeric data. Suppose we suspect a differential equation of the form

$$(a_0 + a_1 x + \cdots + a_{10} x^{10})\, m(x) + (b_0 + b_1 x + \cdots + b_{10} x^{10})\, m'(x) = 0,$$

with unknown integer coefficients a_i, b_i to be determined. So for a specific point x, the task is to find a so-called *integer relation* of the real numbers $m(x), \ldots, x^{10} m(x), m'(x), \ldots, x^{10} m'(x)$. There are well-known algorithms for finding such relations [FB, LLL].

In order to recover the relation from the values at a single point x, we would need to compute these values to a rather high precision, which is not an easy thing to do. We can get along with less precision by using several evaluation points and searching for a simultaneous integer relation of the numbers $m(x), \ldots, x^{10} m(x), m'(x), \ldots, x^{10} m'(x)$, for several x. It turns out that by using enough evaluation points, we just need about 6 decimal digits of accuracy of $m(x)$ and $m'(x)$ for each of these points, in order to establish a convincing guess. Unfortunately, this is a still bit more than what we were able to obtain by a direct computation via transfer matrices.

Supporting Software:

For Maple and C programs, as well as output files, please visit the web-page

`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/onsager.html` .

Acknowledgement.

We thank the anonymous referee for many insightful remarks, that improved our paper considerably.

References

[C] Barry A. Cipra, *An Introduction to the Ising Model*, The American Mathematical Monthly **94** (1987), 937–959.
Available from `http://www.yaroslavvb.com/papers/cipra-introduction.pdf` (accessed May 9, 2018).

[FB] H.R.P. Ferguson and D.H. Bailey, *A Polynomial Time, Numerically Stable Integer Relation Algorithm.* RNR Techn. Rept. RNR-91-032, Jul. 14, 1992.

[KW] H.A. Kramers and G.H. Wannier, *Statistics of the Two-Dimensional Ferromagnet. Part I*, Physical Review **60** (1941), 252-262.
Available from `http://sites.math.rutgers.edu/~zeilberg/akherim/KW1941.pdf` (accessed May 9, 2018).

[LLL] A.K. Lenstra, H.W. Lenstra, and L. Lovasz, *Factoring Polynomials with Rational Coefficients.* Math. Ann. 261, 515-534, 1982.

[O] Lars Onsager, *Crystal Statistics, I. A two-dimensional model with an order-disorder transition*, Physical Reviews **65**(1944), 117–149.
Available from `http://www.phys.ens.fr/~langlois/Onsager43.pdf` (accessed May 9, 2018).

[T] Colin J. Thompson, *"Mathematical Statistical Physics"*, MacMillan, 1972.

[W] Herbert S. Wilf, *What is an Answer?*, The American Mathematical Monthly **89** (1982), 289–292.

[vH] L. van Hove, *Sur L'intgrale de Configuration Pour Les Systmes De Particules Une Dimension*, Physica **16** (1950), 137–143.

[Y] J.M. Yeomans, *"Statistical Mechanics of Phase Transition"*, Oxford University Press, 1992.

Manuel Kauers, Institute of Algebra, J. Kepler University Linz, Altenbergerstrasse 69, 4040 Linz, Austria.
`manuel dot kauers at jku dot at` .

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `DoronZeil at gmail dot com` .

First written: May 22, 2018; This version: Sept. 04, 2018.