A Linear Time and Constant Space Algorithm to Compute the Mixed Moments of the Multivariate Normal Distributions

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Abstract: Using recurrences gotten from the Apagodu-Zeilberger Multivariate Almkvist-Zeilberger algorithm, we present a linear-time and constant-space algorithm to compute the general mixed moments of the k-variate general normal distribution, with any covariance matrix, for any specific k. Besides their obvious importance in statistics, they are also very significant in enumerative combinatorics, since, when the entries of the covariance matrix remain symbolic, they enable us to count in how many ways, in a species with k different genders, a bunch of individuals can all get married, keeping track of the different kinds of the $k(k-1)/2$ possible heterosexual marriages, and the k possible same-sex marriages. We completely implement our algorithm (with an accompanying Maple package, MVNM.txt) for the bivariate and trivariate cases (and hence taking care of our own 2-sex society and a putative 3-sex society), but alas, the actual recurrences for larger k took too long for us to compute. We leave them as computational challenges.

Maple Package

This article is accompanied by a Maple package, `MVNM.txt`, available from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/MVNM.txt` .

The web-page of this article,

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mvnm.html` ,

contains input and output files, referred to in this paper.

The multivariate Normal Distribution

Recall that the *probability density function* (see [T] and [Wik]) of the *multivariate normal distribution* with mean $\mathbf{0}$ and (symmetric) covariance matrix $\mathbf{C} = (c_{ij})_{1 \le i,j \le k}$ is

$$f_{\mathbf{C}}(\mathbf{x}) := \frac{e^{-\frac{1}{2}\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}}}{\sqrt{(2\pi)^k \det \mathbf{C}}} \quad .$$

By simple rescaling we can always assume that all the variances are 1, in other words, that the entries of the main diagonal of $\mathbf{C}$ are all 1.

We are interested in fast computation of the **mixed moments**

$$M_{\mathbf{C}}(m_1, \cdots, m_k) := \int_{R^k} x_1^{m_1} \cdots x_k^{m_k} \, f_{\mathbf{C}}(x_1, \dots, x_k) \, dx_1 \cdots dx_k \quad .$$

One way (not a good one!) to compute these moments, for any specific $(m_1, \dots, m_k)$ is to diagonalize $\mathbf{C}$, make a change of variables and compute an integral of the form

$$\int_{R^k} \prod_{i=1}^{k} \left(\sum_{j=0}^{k} b_{ij} x_j \right)^{m_i} e^{-\frac{1}{2}(x_1^2 + \dots + x_k^2)} \, dx_1 \cdots dx_k \quad .$$

Then expand $\prod_{i=1}^{k} \left(\sum_{j=0}^{k} b_{ij} x_j \right)^{m_i}$ and use the fact that $\int_{-\infty}^{\infty} e^{-x^2/2} x^r \, dx$ is 0 if r is odd and $\sqrt{2\pi} \cdot \frac{r!}{2^{r/2}(r/2)!)}$ if r is even.

A much better way is via the **moment generating function** ([Wik][T])

$$\sum_{0 \le m_1, \dots, m_k < \infty} M_{\mathbf{C}}(m_1, \dots, m_k) \frac{t_1^{m_1} \cdots t_k^{m_k}}{m_1! \cdots m_k!} = e^{\frac{1}{2}(\sum_{1 \le i,j \le k} t_i c_{ij} t_j)} \quad .$$

This is implemented in procedure `MOMd` in the Maple package `MVNM.txt` mentioned above. For example to get the $(3,3,3,3)$-mixed moment for the **generic** four-variate normal distribution, with a general (symbolic) covariance matrix

$$\begin{pmatrix} 1 & c12 & c13 & c14 \\ c12 & 1 & c23 & c24 \\ c13 & c23 & 1 & c34 \\ c14 & c24 & c34 & 1 \end{pmatrix} \quad ,$$

enter

```
lprint(MOMd([[1,c12,c13,c14],[c12,1,c23,c24],[c13,c23,1,c34],[c14,c24,c34,1]],[3,3,3,3]));
```

Defining $M := max(m_1, \dots, m_k)$, this requires $O(M^k)$ time and memory.

Another way is to to use the fact that

$$\int_{R^k} \frac{\partial}{\partial x_1} \left(x_1^{m_1} \cdots x_k^{m_k} \, f_{\mathbf{C}}(x_1, \dots, x_k) \right) dx_1 \cdots dx_k = 0 \quad .$$

Using the product and the chain rule, and expanding, one gets a certain *mixed* recurrence, requiring to compute all the (up to) $m_1 \cdots m_k$ 'previous' values, requiring, again $O(M^k)$ memory and time.

But thanks to the **Apagodu-Zeilberger** [ApZ] multivariate extension of the **Almkvist-Zeilberger** [AlZ] algorithm there exist **pure** recurrences, with polynomial coefficients in $m_1, \dots, m_k$, in **each** of the discrete coordinate directions. The ones for $k = 2$ are fairly simple (they are essentially second-order), but the ones for $k = 3$ are already very complicated. But *once found* (and we did find them!) this enables a linear-time and constant-space algorithm for computing any (m_1, m_2, m_3)-mixed moment. The recurrences are too complicated to be typeset here, but can be read from the Maple source-code of procedure `MOM3` in our Maple package.

The syntax is

`MOM3(c12,c13,c23,[m1,m2,m3]);` .

For example to get the $(10, 10, 10)$ mixed moment as a polynomial in the symbols $c12, c13, c23$, type:

`MOM3(c12,c13,c23,[10,10,10]);` .

This should (and does!) give the same answer as

`MOMd([[1,c12,c13],[c12,1,c23],[c13,c23,1]],[10,10,10]);` .

To really appreciate the superiority of our algorithm, using `MOM3`, over the straightforward `MOM3d`, try, for example

```
restart: read `MVNM.txt`: t0:=time():lu1:=MOM3(1/2,1/3,1/4,[570,560,750]); time()-
t0;
```
,

that would give you the very complicated `lu1` in 2.56 seconds, while

```
t0:=time(): lu2:=MOMd([[1,1/2,1/3],[1/2,1,1/4],[1/3,1/4,1]],[570,560,750]); time()-
t0;
```
,

would confirm that `lu2` and `lu1` are the same (good check!), but it takes 631.007 seconds.

Warning: Don't even try to use floating-point! You would get garbage, due to the complexity of the calculations that accumulate the round-off errors. Both ways would give you erroneous answers unless `Digits` is set very high.

If you keep $c12, c12, c23$ symbolic, the superiority of `MOM3` over `MOMd` is even more apparent.

`restart: read `MVNM.txt`: time(MOM3(c12,c13,c23,[100,50,40]));` , is less than 12 seconds, while doing the same things with `MOMd` takes 100 times longer!

Why this is also Important in Enumerative Combinatorics?

Using what Herb Wilf [Wil] used to call *generatiningfunctionlogy* it is easy to see that, when the entries of the covariance matrix C are kept *symbolic*, then for the bivariate case, the coefficient of c^r in $M_{[[1,c],[c,1]]}(m1, m2)$ is the exact number of ways that $m1$ men and $m2$ women can all get married and there are exactly r heterosexual marriages. The coefficiet of

$$c_{12}^{a_{12}} \, c_{13}^{a_{23}} \, c_{23}^{a_{23}} \quad ,$$

in the polynomial

$$M_{[[1,c_{12},c_{13}],[c_{12},1,c_{23}],[c_{13},c_{23},1]]}(m_1, m_2, m_3) \quad ,$$

is the exact number of ways that, in a 3-gender society, with genders S_1, S_2, S_3, that m_1 individuals of gender S_1, m_2 individuals of gender S_2, and m_3 individuals of gender S_3, can **all** get married (note that you need their total number, $m_1+m_2+m_3$ to be even, or else it is not possible) where there were exactly a_{12} $\{S1,S2\}$ marriages, a_{13} $\{S1,S3\}$ marriages, and a_{23} $\{S2,S3\}$ marriages.

For example, if you want to know the number of ways 300 men and 200 women can get married where there were exactly 100 heterosexual weddings (and hence 150 same-sex marriages), type:

`coeff(MOM2(c,[300,200]),c,100);` ,

to get a certain 564-digit integer.

If you want to know, in a 3-gender society, the **exact** number of ways that 20 individuals of gender S1, 20 individuals of gender S2, and 20 individuals of gender S3, can get married (so altogether there are 30 weddings) with 9 $\{S1,S2\}$ weddings, 7 $\{S1,S3\}$ weddings, and 5 $\{S2,S3\}$ weddings (and hence $30-9-7-5=9$ same-sex marriages), type:

`coeff(coeff(coeff(MOM3(c12,c13,c23,[20,20,20]),c12,9),c13,7),c23,5);` ,

getting, in *0.533* seconds, that the number is:

$$444975998773143505634352562176000000000 \quad .$$

Sample Data

To see the list of lists of lists of polynomials in `c12,c13,c23`, let's call it L, such that

`L[m1][m2][m3]`

is the `(m1,m2,m3)`-mixed moment of the trivariate normal distribution with covariance matrix `[[1,c12,c13],[c12,1,c13],[c13,c23,1]]` for $1 \leq m1, m2, m3 \leq 20$ look at the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMVNM1.txt` .

To see the first 35 diagonal mixed moments (i.e. up to the $(70,70,70)$ mixed moment), see

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMVNM2.txt` .

Enjoy!

The recurrences for four dimensions took too long for us, and we leave them as computational challenges. Perhaps they can be done with Christoph Koutschan's [K] very powerful Mathematica package?

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Exclusively published in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger and arxiv.org

Feb. 20, 2022