Marko Petkovšek (1955-2023), My A=B Mate

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I have only met Marko a few times in my life, but he was a constant presence.

The first time was in 1991, in Philadelphia, shortly after he defended his Carnegie-Mellon PhD under the logical giant Dana Scott (`https://www.mathgenealogy.org/id.php?id=8024`). Herb Wilf invited Marko to speak at UPenn's combinatorics seminar, about the amazing *Petkovšek algorithm*. Both Herb and I were very excited about Marko's algorithm, since that was the *missing link* needed by us to solve the at-the-time *wide open* **decision problem** for the existence of a closed-form solution for **definite hypergeometric summation**.

Recall that the famous Gosper algorithm [G] decides whether an *indefinite* sum

$$F(n) := \sum_{k=1}^{n} f(k) \quad ,$$

where $f(k)$ is a *hypergeometric term* (i.e. $f(k+1)/f(k)$ is **rational function** of k), is again hypergeometric. But what about **definite summation**, i.e.

$$F(n) := \sum_{k=0}^{n} f(n,k) \quad ,$$

where $f(n,k)$ is (proper) hypergeometric in **both** discrete variables n and k? So-called **Wilf-Zeilberger Algorithmic Proof Theory**, and the **Zeilberger algorithm** (see [PWZ]) can always find the next-best thing to a closed-form solution, a **linear recurrence equation with polynomial coefficients**, of *some* **order** L. In other words come up with $L+1$ polynomials: $p_0(n), p_1(n), \dots, p_L(n)$; such that

$$\sum_{i=0}^{L} p_i(n)\, F(n+i) \;=\; 0 \quad .$$

If the order, L, happens to be 1, then we know right away that $F(n)$ is closed form (in the sense of being hypergeometric), but what if $L > 1$?

Zeilberger's algorithm guarantees to output *some* recurrence of *some order*, but does **not** always give you the **minimal order**. In order to know for **sure** that there is no first-order recurrence, we need the amazing **Petkovšek algorithm Hyper** [P1] [P2] [P3] [PWZ] (Ch. 8). That was exactly the missing ingredient that we needed to settle this important **decision problem**.

As Marko describes so charmingly in his reminiscences about Herb Wilf [CGH], this connection lead to our collaboration A=B [PWZ], whose *table of contents* was drafted in the Moosewood vegetarian restaurant, in Itacha, New York.

Marko then went on to become a leader in **symbolic summation** and **difference equations**, with very deep work in collaboration with Sergei A. Abramov, and others. He also wrote a charming paper with Herb Wilf about a 'high-tech' proof of an important identity in enumerative combinatorics [PW].

When Bill Chen organized a conference in Tianjin to celebrate my 60^{th} birthday, in the summer of 2010, he asked me whom to invite, and of course I suggested Marko, whom Bill dully invited. Unfortunately, Marko was unable to come, but instead wrote me this nice email.

`From Marko.Petkovsek@fmf.uni-lj.si Sun Aug 8 17:30:52 2010`

`Dear Doron,`

`I was looking forward very much to the Zeilberger-fest at Nankai. Alas, as it turned out, I will not be able to attend it. Last week I had a small operation on my leg (nothing serious - removal of a large carbuncle probably caused by an insect bite), but the wound has not healed yet and my doctor advised against traveling.`

`So, I just wish you a very happy` $|A_5|$`-th birthday!`

`Best regards,`

`--Marko`

Last time I met Marko in person was during Herb Wilf's 80^{th} Birthday conference, held May 26-29, 2011, in Wilfrid Laurier University, Waterloo, Canada, and organized by Eugene Zima and Ilias Kotsireas `https://cargo.wlu.ca/W80/`. Marko gave a great talk on *enumeration of structures with no forbidden substructures* that clearly showed that he had a very broad perspective of combinatorial enumeration. In his own words
(from `https://sites.math.rutgers.edu/~zeilberg/akherim/W80abstracts.pdf`):

Many interesting classes of combinatorial structures are defined by restricting some general class of structures to those structures that avoid certain forbidden substructures. Examples include words avoiding forbidden subwords or subsequences, permutations avoiding forbidden patterns, matrices avoiding forbidden submatrices, graphs avoiding forbidden subgraphs, induced subgraphs, minors, or topological minors. We will try to look at the abundance of enumeration problems (solved and unsolved) presented by such classes.

Marko will be sorely missed, but his mathematics and algorithms guarantee his **immortality**.

References

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[P1] Marko Petkovšek, *Finding closed-form solutions of difference equations by symbolic methods*, PhD. thesis, Carnegie-Mellon University, CMU0-CS-91-103, 1991.
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