Efficient Weighted Counting of Multiset Derangements

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Dedicated to Mourad El-Houssieny ISMAIL (born April 27, 1944), with friendship and admiration

Abstract: We use the Almkvist-Zeilberger algorithm, combined with a weighted version of the Even-Gillis Laguerre integral due to Foata and Zeilberger, in order to efficiently compute weight enumerators of multiset derangements according to the number of cycles. The present paper is inspired by important previous work by Mourad Ismail and his collaborators, done in the late 1970s, but still useful after all these years.

Multiset Derangements

The (generalized) Laguerre polynomials, $L_k^{(\alpha)}(x)$, are defined as follows:

$$L_k^{(\alpha)}(x) := \sum_{i=0}^{n} (-1)^i \frac{(\alpha+i+1)(\alpha+i+2)\dots(\alpha+n)}{i!(n-i)!} x^i \quad .$$

Let $A_1, A_2, \dots, A_n$ be n pairwise-disjoint sets of cardinalities $k_1, \dots, k_n$ respectively. A permutation π of $A := A_1 \cup \dots \cup A_n$ is a multiset derangement if for every $1 \le i \le n$, whenever $x \in A_i$, $\pi(x) \notin A_i$. Let $D(k_1, \dots, k_n)$ be the set of such multiset derangements.

As usual, for any finite set S, let $|S|$ denote its number of elements.

Shimon Even and Joe Gillis [EvG] (see also [As], [G], and [Z]) proved that

$$|D(k_1, \dots, k_n)| = (-1)^{k_1+\dots+k_n} \left(\prod_{i=1}^{n} k_i! \right) \int_0^\infty \left(\prod_{i=1}^{n} L_{k_i}^{(0)}(x) \right) e^{-x}\, dx \quad . \tag{1}$$

Comment: Usually the elements of each A_i are identified, and the formula then does not have $(\prod_{i=1}^{n} k_i!)$ in front, but for the purpose of the present paper, as was done in [FZ], we have $k_1+\dots+k_n$ distinct elements, all preserving their individuality.

For a permutation π, let $cyc(\pi)$ denote its *number of cycles*. For example $cyc(1234) = 4$, and $cyc(2341) = 1$. Recall that, famously,

$$\sum_{\pi \in S_n} \alpha^{cyc(\pi)} = \alpha(\alpha+1)\cdots(\alpha+n-1) \quad .$$

This is easily proved by considering how to form permutations of $\{1, ..., n\}$ out of those of $\{1, ..., n-1\}$. Denote the left side by $f_n(\alpha)$. Given a permutation of $\{1, ..., n-1\}$ we can insert n inside any of the existing cycles, and

there are $n-1$ ways of doing it, preserving the number of cycles, or else create a brand-new cycle with n alone, increasing the number of cycles by one. Hence $f_n(\alpha) = ((n-1)+\alpha) f_{n-1}(\alpha)$.

Let $A(k_1, \ldots, k_n)(\alpha)$ be the weight enumerator, according to the number of cycles, of the set of multiset derangements, $D(k_1, \ldots, k_n)$, in other words

$$A(k_1, \ldots, k_n)(\alpha) := \sum_{\pi \in D(k_1, \ldots, k_n)} \alpha^{cyc(\pi)} \quad .$$

In 1988, Dominique Foata and one of us (DZ) [FZ], inspired by the work of Mourad Ismail and his collaborators ([AsIs], [AsIsRa], [AsIsKo]) proved the following α-analog of the Even-Gillis theorem:

$$A(k_1, \ldots, k_n)(\alpha) = \frac{(-1)^{k_1+\ldots+k_n}}{(\alpha-1)!} \left(\prod_{i=1}^{n} k_i! \right) \int_0^\infty \left(\prod_{i=1}^{n} L_{k_i}^{(\alpha-1)}(x) \right) x^{\alpha-1} e^{-x} \, dx \quad . \tag{2}$$

Note that this is a polynomial of degree $(k_1 + \ldots + k_n)/2$ rather than $k_1 + \ldots + k_n$, since every cycle is at least of length 2.

In this paper we will focus on efficient computations of many terms of the sequences $A(k, k, \ldots, k)(\alpha)$, where k is repeated n times, for specific (small, and not so small) k, but arbitrarily large n. In other words, our goal is to compute as many as possible terms of the sequences

$$F_k(n)(\alpha) := A(k, \ldots, k)(\alpha) \quad , \quad (k \quad repeated \quad n \quad times) \quad ,$$

for $k = 1$, $k = 2$, etc.

By (2) we have

$$F_k(n)(\alpha) = \frac{(-1)^{kn} k!^n}{(\alpha-1)!} \int_0^\infty (L_k^{(\alpha-1)}(x))^n x^{\alpha-1} e^{-x} \, dx \quad . \tag{3}$$

Using the **Almkvist-Zeilberger algorithm** [AlZ] (see [D] for a lucid and engaging account), one of us (SBE), using our Maple package `Mourad.txt`, available from the front of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mourad.html` ,

discovered (and proved) the following recurrences for $F_k(n) = F_k(n)(\alpha)$ for $1 \leq k \leq 10$. They get increasingly complicated, and we will only state the first two in the body of this paper.

$$-\alpha\,(n+1)\,F_1(n) - (n+1)\,F_1(n+1) + F_1(n+2) = 0 \quad . \tag{4}$$

$$4\alpha\,(2n+5)\,(n+2)\,(n+1)\,(\alpha+1)^2\,F_2(n)$$
$$+2\,(n+2)\,(\alpha+1)\left(4\alpha\,n^2+12\alpha n-4n^2+7\alpha-14n-10\right)F_2(n+1)$$
$$-2\,(n+2)\left(4\alpha n+4n^2+8\alpha+16n+17\right)F_2(n+2)+(2n+3)\,F_2(n+3)=0 \quad . \tag{5}$$

For linear recurrences for $F_k(n)$ for $3 \leq k \leq 10$ see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMourad1.txt` .

We observe that the order of the recurrence for $F_k(n)$ is $k+1$ and its degree in n is $\binom{k+1}{2}$.

These recurrences enable very fast computation of quite a few terms of these sequences. It is all implemented in the already mentioned Maple package `Mourad.txt`. The direct url of this package is:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/Mourad.txt` .

A User's Manual to the Maple package Mourad.txt

To use it first download it to your favorite directory (usually Downloads). Start a Maple worksheet, make sure that the directory is the right one, and then type

`read 'Mourad.txt' ;` .

To get a list of the procedures, type `ezra();` . The main procedures are:

• `Wder(L,a)`, that inputs a list of positive integers, $L=[k_1,\dots,k_n]$ and implements Equation (2). This is very slow, and should not be used for large L.

• `Operk(k,a,n,N)`, that inputs a positive integer k, and outputs the recurrence operator (where N is the shift operator in n) annihilating the sequence $F_k(n)$. These get very complicated for human eyes, but the computer does not mind and it enables a fast computation of many terms.

• `SeqkF(k,K,`α`)`: inputs a positive integer k and a positive integer K and a symbol α and outputs the first K terms of the sequence of polynomials $F_k(n)=F_k(n)(\alpha)$. For example, to get the weight enumerator of the set of permutations of a standard deck of cards where no card can wind up at a location that originally was occupied by the same number (1 through 13, where Jack, Queen, and King stand for $11,12,13$ respectively) but it is OK to have the same suit, type:

`SeqkF(4,13,`α`)[13];` ,

and you would get, in a few seconds, the following polynomial in α of degree 26

$$626486325682388256883179081695232\,\alpha^{26}+$$
$$3948815860811007759557670403206807552\,\alpha^{25}+$$
$$42261604469281014106759334470421934243 84\,\alpha^{24}+$$
$$1829313185198525509532452983498671376039936\,\alpha^{23}+$$
$$42595522713357731227339242131006802911821 8240\,\alpha^{22}+$$
$$6156871138225571569934341483286576175279 5578368\,\alpha^{21}+$$
$$6015599331237497842549834616372527226200006852608\,\alpha^{20}+$$
$$4200305131029962895456184953183553479685792396738 56\,\alpha^{19}+$$
$$2177938552960678830806506675243564165556602703079 0144\,\alpha^{18}+$$
$$8619310094635805651425154543519244755566038025764 86400\,\alpha^{17}+$$
$$2655692681177260330693451189378249830933081179240 0580608\,\alpha^{16}+$$
$$6462194193866020459078242285766825278512060565544847278 08\,\alpha^{15}+$$
$$1254416614780840084133408162808155401873966239427260422 5536\,\alpha^{14}+$$
$$1955254085465389126903782512876804882197929430922129194 35264\,\alpha^{13}+$$
$$2455695605166443718371007842011087818790955115435503879454720\,\alpha^{12}+$$
$$2486704814667222730934566698991379670472881012675282002037 9648\,\alpha^{11}+$$
$$2025697939116132741829290190821850922610141572010853691534 86848\,\alpha^{10}+$$
$$1320388339665569428585764027609539765653334771119656423470923776\,\alpha^{9}+$$
$$6825167923093955037138102373992833000704975000443998456569135104\,\alpha^{8}+$$
$$2760280932892183531379368206812130371214230427009961182130864 1280\,\alpha^{7}+$$
$$8564734270599332214814823577740100744793222360715969121098504 6016\,\alpha^{6}+$$
$$1981596638309000440426417890392538651226170202300653970806024437 76\,\alpha^{5}+$$
$$3275474736857246875871889950327146249999306890307177019801201541 12\,\alpha^{4}+$$
$$3611482150045173124936459005174448441688597747240705027687401390 08\,\alpha^{3}+$$
$$2344260654005149769534175247988119027071099693816953194471961395 20\,\alpha^{2}+$$
$$6639494805094683093248405826364448867272260835506705561959792640 0\,\alpha \quad .$$

Plugging-in $\alpha = 1$ and dividing by $4!^{13}$ (to get derangements where members of the same set are identified) gives you:

$$1493804444499093354916284290188948031229880469556 \quad ,$$

agreeing with the title of [EkKZ], that handled the *straight*, rather than the *weighted*, enumeration of multiset derangements.

Note that the largest coefficient in the above polynomial is that of a^3, hence the *mode* of the random variable 'number of cycles' in a random multi-set derangement of a standard deck is 3. The average happens to be $3.586337835\ldots$ while the standard-deviation (the squre-root of the variance) is $1.412546929\ldots$

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References

[AlZ] Gert Almkvist and Doron Zeilberger, *The method of differentiating under the integral sign*, J. Symbolic Computation **10**, 571-591 (1990).
`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/duis.html` .

[As] Richard Askey, *"Orthogonal Polynomials and Special Functions"*, SIAM, 1975.

[AsIs] R. Askey and M.E.H. Ismail, *Permutation problems and special functions*, Canad. J. Math. **28** (1976), 853-874.

[AsIsKo] R.Askey, M.E.H. Ismail, and T. Koornwinder, *Weighted permutation problems and Laguerre polynomials*, J. Comb. Theory (Ser. A.) **25** (1978), 277-287.

[AsIsRa] R.Askey, M.E.H. Ismail, and M.T. Rashed, *A derangement problem.* Technical report 1522, UW-Madison Mathematics Research Center, June 1975.

[D] Robert Dougherty-Bliss, *Integral Recurrences from A to Z*, The American Mathematical Monthly **129** (2021), 805-815.

[EkKZ] Shalosh B. Ekhad, Christoph Koutschan, and Doron Zeilberger, *There are EXACTLY* 1493804444499093354916284290188948031229880469556 *ways to derange a standard deck of cards (ignoring suits) [and many other such useful facts]*, Enumerative Combinatorics and Applications **1** (2021), issue 3, article S2A17.
`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/multider.html` .

[EvG] S. Even and J. Gillis, *Derangements and Laguerre polynomials*, Math. Proc. Camb. Phil. Soc. **79** (1976), 135-143.

[FZ] Dominique Foata and Doron Zeilberger, *Laguerre polynomials, weighted derangements, and positivity*, SIAM J. Discrete Mathematics **1** (1988), 425-433.

[G] J. Gillis, *The statistics of derangement-a survey*, Journal of Statistical Physics **58** (1990),575-578.

[GIO] J. Gillis, M. E. H. Ismail, and T. Offer, *An Asymptotic Problem in Derangement Theory*, SIAM Journal on Mathematical Analysis **21** (1990), 262-269.

[Z] Doron Zeilberger, *How Joe Gillis discovered combinatorial special function theory*, Math. Intell. **17(2)** (Spring 1995), 65-66.
https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gillis.html

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