Efficient Weighted Counting of Multiset Derangements

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Dedicated to Mourad El-Houssieny ISMAIL (born April 27, 1944), with friendship and admiration

Abstract: We use the Almkvist-Zeilberger algorithm, combined with a weighted version of the Even-Gillis Laguerre integral due to Foata and Zeilberger, in order to efficiently compute weight enumerators of multiset derangements according to the number of cycles. The present paper is inspired by important previous work by Mourad Ismail and his collaborators, done in the late 1970s, but still useful after all these years.

Multiset Derangements

The (generalized) Laguerre polynomials, $L_k^{(\alpha)}(x)$, are defined as follows:

$$L_k^{(\alpha)}(x) := \sum_{i=0}^{n} (-1)^i \, \frac{(\alpha+i+1)(\alpha+i+2)\dots(\alpha+n)}{i!(n-i)!} \, x^i \quad .$$

Let $A_1, A_2, \dots, A_n$ be n pairwise-disjoint sets of cardinalities $k_1, \dots, k_n$ respectively. A permutation π of $A := A_1 \cup \dots \cup A_n$ is a multiset derangement if for every $1 \le i \le n$, whenever $x \in A_i$, $\pi(x) \notin A_i$. Let $D(k_1, \dots, k_n)$ be the set of such multiset derangements.

As usual, for any finite set S, let $|S|$ denote its number of elements.

Shimon Even and Joe Gillis [EvG] (see also [As], [G], and [Z]) proved that

$$|D(k_1, \dots, k_n)| = (-1)^{k_1+\dots+k_n} \left(\prod_{i=1}^{n} k_i! \right) \int_0^\infty \left(\prod_{i=1}^{n} L_{k_i}^{(0)}(x) \right) e^{-x} \, dx \quad . \tag{1}$$

Comment: Usually the elements of each A_i are identified, and the formula then does not have $(\prod_{i=1}^n k_i!)$ in front, but for the purpose of the present paper, as was done in [FZ], we have $k_1 + \dots + k_n$ distinct elements, all preserving their individuality.

For a permutation π, let $cyc(\pi)$ denote its *number of cycles*. For example $cyc(1234) = 4$, and $cyc(2341) = 1$. Recall that, famously,

$$\sum_{\pi \in S_n} \alpha^{cyc(\pi)} \, = \, \alpha(\alpha+1)\cdots(\alpha+n-1) \quad .$$

This is easily proved by considering how to form permutations of $\{1, ..., n\}$ out of those of $\{1, ..., n-1\}$. Denote the left side by $f_n(\alpha)$. Given a permutation of $\{1, ..., n-1\}$ we can insert n inside any of the existing cycles, and there are $n-1$ ways of doing it, preserving the number of cycles, or else create a brand-new cycle with n alone, increasing the number of cycles by one. Hence $f_n(\alpha) = ((n-1)+\alpha) f_{n-1}(\alpha)$.

Let $A(k_1, \dots, k_n)(\alpha)$ be the weight enumerator, according to the number of cycles, of the set of multiset derangements, $D(k_1, \dots, k_n)$, in other words

$$A(k_1, \dots, k_n)(\alpha) := \sum_{\pi \in D(k_1, \dots, k_n)} \alpha^{cyc(\pi)} \quad .$$

In 1988, Dominique Foata and one of us (DZ) [FZ], inspired by the work of Mourad Ismail and his collaborators ([AsIs], [AsIsRa], [AsIsKo]) proved the following α-analog of the Even-Gillis theorem:

$$A(k_1, \dots, k_n)(\alpha) = \frac{(-1)^{k_1+\dots+k_n}}{(\alpha-1)!} \left(\prod_{i=1}^{n} k_i! \right) \int_0^\infty \left(\prod_{i=1}^{n} L_{k_i}^{(\alpha-1)}(x) \right) x^{\alpha-1} e^{-x} \, dx \quad . \tag{2}$$

Note that this is a polynomial of degree $(k_1 + \dots + k_n)/2$ rather than $k_1 + \dots + k_n$, since every cycle is at least of length 2.

In this paper we will focus on efficient computations of many terms of the sequences $A(k, k, \dots, k)(\alpha)$, where k is repeated n times, for specific (small, and not so small) k, but arbitrarily large n. In other words, our goal is to compute as many as possible terms of the sequences

$$F_k(n)(\alpha) := A(k, \dots, k)(\alpha) \quad , \quad (k \quad \textit{repeated} \quad n \quad \textit{times}) \quad ,$$

for $k = 1$, $k = 2$, etc.

By (2) we have

$$F_k(n)(\alpha) = \frac{(-1)^{kn}(kn)!}{(\alpha-1)!} \int_0^\infty (L_k^{(\alpha-1)}(x))^n x^{\alpha-1} e^{-x} \, dx \quad . \tag{3}$$

Using the **Almkvist-Zeilberger algorithm** [AlZ] (see [D] for a lucid and engaging account), one of us (SBE), using our Maple package `Mourad.txt`, available from the front of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mourad.html` ,

discovered (and proved) the following recurrences for $F_k(n) = F_k(n)(\alpha)$ for $1 \leq k \leq 10$. They get increasingly complicated, and we will only state the first two in the body of this paper. From now on a will stand for α.

$$-a\,(n+1)\,F_1(n) - (n+1)\,F_1(n+1) + F_1(n+2) = 0 \quad . \tag{4}$$

$$4a\,(2n+5)\,(n+2)\,(n+1)\,(a+1)^2\,F_2(n)$$
$$+2\,(n+2)\,(a+1)\left(4a\,n^2+12an-4n^2+7a-14n-10\right)F_2(n+1)$$
$$-2\,(n+2)\left(4an+4n^2+8a+16n+17\right)F_2(n+2)+(2n+3)\,F_2(n+3)=0 \quad . \tag{5}$$

For linear recurrences for $F_k(n)$ for $3 \le k \le 10$ see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMourad1.txt` .

These recurrences enable very fast computation of quite a few terms of these sequences. It is all implemented in the already mentioned Maple package `Mouard.txt`. The direct url of this package is:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/Mourad.txt` .

A User's Manual to the Maple package Mourad.txt

To use it first download it to your favorite directory (usually Downloads). Start a Maple worksheet, make sure that the directory is the right one, and then type

`read ‘Mouard.txt‘ ;` .

To get a list of the procedures, type `ezra();` . The main procedures are:

• `Wder(L,a)`, that inputs a list of positive integers, $L = [k_1, \dots, k_n]$ and implements Equation (2). This is very slow, and should not be used for large L.

• `Operk(k,a,n,N)`, that inputs a positive integer k, and outputs the recurrence operator (where N is the shift operator in n) annihilating the sequence $F_k(n)$. These get very complicated for human eyes, but the computer does not mind and it enables a fast computation of many terms.

• `SeqkF(k,K,a)`: inputs a positive integer k and a positive integer K and a symbol a (what we called α above) and outputs the first K terms of the sequence of polynomials $F_k(n) = F_k(n)(a)$. For example, to get the weight enumerator of the set of permutations of a standard deck of cards where no card can wind up at a location that originally was occupied by the same number (1 through 13, where Jack, Queen, and King stand for $11, 12, 13$ respectively) but it is OK to have the same suit, type:

`SeqkF(4,13,a)[13];` ,

and you would get, in a few seconds, the following polynomial of degree 26

$$626486325682388256883179081695232\,a^{26}+$$

$$3948815860811007759557670403206807552\,a^{25}+$$

$$4226160446928101410675933447042193424384\,a^{24}+$$

$$182931318519852550953245298349867137603 9936\,a^{23}+$$

$$42595522713357731227339242131006802911821824 0\,a^{22}+$$

$$61568711382255715699343414832865761752795578368\,a^{21}+$$

$$601559933123749784254983461637252722620000685 2608\,a^{20}+$$

$$42003051310299628954561849531835534796857923967 3856\,a^{19}+$$

$$21779385529606788308065066752435641655566027030790144\,a^{18}+$$

$$861931009463580565142515454351924475556603802576486400\,a^{17}+$$

$$2655692681177260330693451189378249830933081179240058 0608\,a^{16}+$$

$$64621941938660204590782422857668252785120605655448472 7808\,a^{15}+$$

$$125441661478084008413340816280815540187396623942726042 25536\,a^{14}+$$

$$19552540854653891269037825128768048821979294309221291943 5264\,a^{13}+$$

$$24556956051664437183710078420110878187909551154355038794 54720\,a^{12}+$$

$$248670481466722273093456669899137967047288101267528200203 79648\,a^{11}+$$

$$202569793911613274182929019082185092261014157201085369153 486848\,a^{10}+$$

$$132038833966556942858576402760953976565333477111965642347 0923776\,a^{9}+$$

$$682516792309395503713810237399283300070497500044399845656 9135104\,a^{8}+$$

$$2760280932892183531379368206812130371214230427009961182130 8641280\,a^{7}+$$

$$8564734270599332214814823577740100744793222360715969121098 5046016\,a^{6}+$$

$$19815966383090004404264178903925386512261702023006539708060 2443776\,a^{5}+$$

$$32754747368572468758718899503271462499993068903071770198012 0154112\,a^{4}+$$

$$36114821500451731249364590051744484416885977472407050276874 0139008\,a^{3}+$$

$$23442606540051497695341752479881190270710996938169531944719 6139520\,a^{2}+$$

$$66394948050946830932484058263644488672722608355067055619597 926400\,a \quad .$$

Plugging-in $a=1$ and dividing by $4!^{13}$ (to get derangements where members of the same set are identified), we get

$$149380444449909335491628429018894803122988046 9556 \quad ,$$

agreeing with the title of [EkKZ], that handled the *straight*, rather than the *weighted*, enumeration of multiset derangements.

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