In How Many Ways Can a King Return Home After n Steps?

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Theorem. If $f(n)$ is the number of ways a Chess King can walk n steps, returning to the starting point, in an "infinite" chessboard, then $f(0) = 1, f(1) = 0, f(2) = 8$ (obviously!) and for $n \geq 0$

$$f(n+3) =$$

$$\frac{(3n+5)(n+2)(3n+8)}{(n+3)^2(4+3n)} f(n+2) + \frac{4\left(27n^3+144n^2+248n+139\right)}{(n+3)^2(4+3n)} f(n+1) + \frac{32(3n+7)(n+1)^2}{(n+3)^2(4+3n)} f(n) \quad .$$

We also have the asymptotic formula:

$$f(n) = \frac{2}{3\pi}\frac{8^n}{n}(1 - 4/9\,n^{-1} + 1/18\,n^{-2} + \frac{29}{486}n^{-3} + \frac{445}{17496}n^{-4} - \frac{443}{8748}n^{-5} - \frac{10933}{104976}n^{-6} +$$

$$\frac{35761}{944784}n^{-7} + \frac{3502795}{7558272}n^{-8} + \frac{13332763}{38263752}n^{-9} - \frac{30042779573}{11019960576}n^{-10}) + O(n^{-12}) \quad .$$

Proof. Since *combinatorics is algebra*, and conversely, *algebra is combinatorics*, $f(n)$ is the coefficient of x^0y^0 in the polynomial $(x + 1/x + y + 1/y + xy + 1/(xy) + x/y + y/x)^n$, that can be expressed as a (formal!) double-contour integral, that is beautifully handled by Moa Apagodu and Doron Zeilberger's Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/MultiAlmkvistZeilberger` ,

that is explained in their article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/multiZ.pdf` .

Once we have the recurrence, the asymptotic formula was derived using the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/AsyRec` ,

that is briefly explained in Doron Zeilberger's article:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/asy.pdf` .

Everything is rigorous *except* for the constant in front, $\frac{2}{3\pi}$. It was found (empirically) by first estimating the constant in front (divide $f(n)$ by the expression on the right (except for the constant), for large n, say $n = 1000$, and then getting a numerical estimate for that constant, and then *identifying* it using Maple's built-in command `identify`.)

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Of course, that constant can be easily rigorously obtained in other ways. One way is to convert the double contour integral into a double trigonometric integral, and use standard methods, and another way is to use probability (find the asymptotic covariance matrix, and use the local limit law) , but who cares? □

For the convenience of the readers, all the necessary tools have been assembled into **one** self-contained Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/WalkPapers` ,

that can produce as many papers like this one as desired. Some other results are posted here:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/melech.html` .

Comment. The first few terms of the sequence (staring with $n = 1$) are

$$0\,,\,8\,,\,24\,,\,216\,,\,1200\,,\,8840\,,\,58800\,,\,423640\,,\,3000480\,,\,21824208\,,\,158964960\,,\,1171230984\,,\,\ldots.$$

This is Sloane's `http://oeis.org/A098070`, where one can find a *fourth*-order, and hence, not as good as our *third*-order, linear recurrence. As with many entries in Sloane's OEIS, one can't tell whether the recurrence is only *guessed* or actually *proved*. Of course, thanks to WZ theory, it is known, *a priori*, in many cases, that a linear recurrence (with polynomial coefficients) *exists*, and hence we have a semi-rigorous meta-proof, that could be made completely rigorous, that the guessed recurrence is *provably* correct, but it is (often) easier to use WZ (or multi-WZ) *ab initio*, because the later method gives you a recurrence, **and its proof**, at the *same time*.

A Quick Guide to the Maple package `WalkPapers`

Once you have downloaded `WalkPapers` to your current directory, get into Maple, and type

`read WalkPapers:` .

For a list of the procedures, type `ezra();`, and for help with any specific procedure, type `ezra(ProcedureName)`.

The main procedure is `WalkPaper(S,n,m,K1,K2)`, where

(i) `S` is the set of steps (a set of lists of integers of the same size)

(ii) `n` is a symbol

(iii) `m` is a positive integer indicating the desired order for the asymptotic formula for $f(n)$.

(iv) $K1$ is a positive integer, indicating how many terms of the sequence you would like to have displayed.

(v) $K2$ is a large (recommended at least 1000) that is used to estimate, and `identify` the constant in front of the asymptotic formula.

For example, the present article was gotten by typing:

`WalkPaper({[1,0],[-1,0],[0,1],[0,-1],[1,-1],[-1,1],[1,1],[-1,-1]},n,10,30,1000):` .

`WalkPaperSR(S,n,m,K1,K2, MaxC)` is a semi-rigorous analog, that sometimes works faster. (See the *ezra* for the meaning of `MaxC`.)

`Sefer1D` and `Sefer2D` produce webbooks with lots of theorems for different sets of steps in one and two dimensions, respectively.

Encore: Knight's Walks

For a third-order linear recurrence, and an asymptotic formula, for the number of ways for a Knight to return home after $2n$ steps (in an infinite chessboard), see:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oWalkPapersKnightSR` .

The first few terms (starting with $n = 1$) are:

$8, 168, 5840, 261800, 13180608, 702273264, 38641656768, 2171652448680, 123938999632448, 7158206751686848, \ldots$