A Loving Rendition of the Marcus-Tardos Amazing Proof of the Füredi-Hajnal Conjecture

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There are n men and n women, all of different handsomeness and beauty respectively, and m (heterosexual) *love affairs* ($0 \leq m \leq n^2$). Fix k and a permutation π of size k. We want to avoid the *love-pattern* π, i.e. we forbid that there exist k men and k women such that, for $i = 1, \ldots, k$, the i-th most handsome man (amongst these k men) loves the $\pi(i)$-th prettiest woman (amongst these k women). **Zoltán Füredi** and **Péter Hajnal** (**Discrete Math 103 (1992), 233-251**) conjectured that $m \leq C_k \cdot n$, for some constant C_k. **Adam MARCUS** and **Gábor TARDOS** (preprint, available from **http://www.renyi.hu/~tardos/**) have just proven it. I love their proof so much that I decided to rephrase it in the evocative language of **love**.

Fix a love-pattern π of size k, and let $f(n)$ be the maximum number of love affairs for which you can still avoid π. Let n be divisible by a. Partition the men and the women into n/a clubs where the a most handsome men belong to the first club, the next a most handsome men belong to the second club, etc. and ditto for the women. A club *Loves* a club of the opposite sex if there is at least one love affair between their members. A club *Adores* a club of the opposite sex if there are at least k members of the former club who love someone in the latter club. The total number of Loving Pairs amongst the n/a men- and women- clubs is $\leq f(n/a)$ (or else a "meta-pattern" would yield a pattern). A club can Adore at most $k\binom{a}{k}$ clubs of the opposite sex, since otherwise (**pigeonhole!**) there would be $\geq k$ Adored clubs who get Adored thanks to the *same* k members of *one* of the $\binom{a}{k}$ k-subsets of the Adoring club, thereby containing *any* love-pattern of size k, in particular π. Hence there are $\leq k\binom{a}{k} \cdot 2\frac{n}{a}$ Adoring Pairs (of clubs, from either end). The number of love affairs between members of Loving but non-Adoring clubs is $\leq (k-1)^2$ (or else there would be at least k persons in one of these two clubs who love people in the other club), and the number of love affairs between Adoring clubs is (trivially) $\leq a^2$. This implies the **divide-and conquer** inequality $f(n) \leq (k-1)^2 \cdot f(\frac{n}{a}) + 2ak\binom{a}{k} \cdot n$, that entails $f(n) \leq \frac{2a^2k\binom{a}{k}}{a-(k-1)^2} \cdot n$. Now take any $a \geq (k-1)^2 + 1$. Marcus and Tardos took $a = k^2$, getting $C_k = 2k^4\binom{k^2}{k}$. $\square$

Thanks to **Martin Klazar** (**FPSAC 2000, 250-255**), the **Stanley-Wilf** conjecture is an immediate corollary. Let $F(n)$ be the number of love-configurations avoiding π. Now partition the men and women into "clubs" of size 2. We have $F(n) \leq F(n/2) \cdot 15^{C_k \cdot n/2}$ (deciding the love-configuration for the clubs, and which of the $2^4 - 1$ love-configurations between each of the $\leq C_k \cdot n/2$ Loving Pairs). This implies $F(n) \leq 15^{C_k \cdot n}$, hence $F(n)$ is of exponential growth.

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