# THE MAHONIAN PROBABILITY DISTRIBUTION ON WORDS IS ASYMPTOTICALLY NORMAL

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Abstract. We study XXX

[ Abstract! Correct date! No showkeys! ] xxx

S:intro

## 1. Introduction

The most important discrete probability distribution, by far, is the *Binomial distribution*, $B(n,p)$ for which we know everything *explicitly*, $\mathbb{P}(X=i)$ $(\binom{n}{i}p^i(1-p)^{n-i})$, the probability generating function $((pt+(1-p))^n)$, the moment generating function $((pe^t+1-p)^n)$, etc. etc. Most importantly, it is *asymptotically normal*, which means that the normalized random variable

$$Z_n = \frac{X_n - np}{\sqrt{np(1-p)}}$$

tends to the standard Normal distribution $N(0,1)$, as $n \to \infty$.

Another important discrete distribution function is the *Mahonian* distribution, defined on the set of *permutations* on $n$ objects, and describing, inter-alia, the random variable "number of inversions". (Recall that an inversion in a permutation $\pi_1, \dots, \pi_n$ is a pair $1 \le i < j \le n$ such that $\pi_i > \pi_j$). Let us call this random variable $M_n$. The probability generating function, due to Netto, is given *explicitly* by:

$$F_n(q) = \frac{1}{n!}\prod_{i=1}^{n}\frac{1-q^i}{1-q} \quad . \tag{1.1}$$ netto

The formula (1.1) has a simple probabilistic interpretation (see Feller's account in [3, Section X.6]): If $Y_j$ is the number of $i$ with $1 \le i < j$ and $\pi_i > \pi_j$, then

$$M_n = Y_1 + \dots + Y_n, \tag{1.2}$$ nettoy

and $Y_1, \dots, Y_n$ are independent random variables and $Y_j$ is uniformly distributed on $\{0, \dots, j-1\}$, as is easily seen by constructing $\pi$ by inserting

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Accompanied by Maple package MahonianStat available from
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mahon.html`. The work of D. Zeilberger was supported in part by the United States of America National Science Foundation.

$1, \dots, n$ in this order at random positions; thus $Y_j$ has probability generating function $(1-q^j)/(j(1-q))$. It follows from (1.1) or (1.2) by simple calculations that the Mahonian distribution has mean and variance

$$\mathbb{E}\, M_n = \frac{n(n-1)}{4}, \qquad (1.3) \quad \boxed{\texttt{em}}$$

$$\operatorname{Var} M_n = \frac{n(n-1)(2n+5)}{72} = \frac{2n^3+3n^2-5n}{72}. \qquad (1.4) \quad \boxed{\texttt{varm}}$$

Even though there is no explicit expression for the coefficients themselves (i.e. for the exact probabilitity that a permutation of $n$ objects would have a certain number of inversions), it is a classical result (see [3, Section X.6]), that follows from an extended form of the Central Limit Theorem, that the normalized version

$$\frac{M_n - n(n-1)/4}{\sqrt{(2n^3+3n^2-5n)/72}} \quad ,$$

tends to $N(0,1)$, as $n \to \infty$. So this sequence of probability distributions, too, is asymptotically normal.

But what about *words*, also known as *multi-set permutations*?. Permutations on $n$ objects can be viewed as words in the alphabet $\{1, 2, \dots, n\}$, where each letter shows up *exactly* once. But what if we allow *repetitions*? I.e., we consider all words with $a_1$ occurrences of 1, $a_2$ occurrences of 2, ..., $a_m$ occurrences of $m$. (We assume throughout that $m \geq 2$ and each $a_j \geq 1$.) We all know that the number of such words is the multinomial coefficient

$$\binom{a_1 + \cdots + a_m}{a_1, \dots, a_m}$$

and many of us also know that the number of such words with exactly $k$ inversions is the coefficient of $q^k$ in the $q$-analog of the multinomial coefficient

$$\binom{a_1 + \cdots + a_m}{a_1, \dots, a_m}_q := \frac{[a_1 + \cdots + a_m]!}{[a_1]! \cdots [a_m]!} \quad , \qquad (1.5) \quad \boxed{\texttt{qmulti}}$$

where $[n]! := [1][2] \cdots [n]$, and $[n] := (1-q^n)/(1-q)$. Assuming that all words are equally likely (the uniform distribution), the probability generating function is

$$F_{a_1,\dots,a_m}(q) := \frac{(\prod_{i=1}^m a_i!) \cdot \prod_{i=1}^{a_1+\cdots+a_m}(1-q^i)}{(a_1+\cdots+a_m)! \prod_{j=1}^m \prod_{i=1}^{a_j}(1-q^i)} = \frac{F_{a_1+\cdots+a_m}(q)}{F_{a_1}(q) \cdots F_{a_m}(q)}. \qquad (1.6) \quad \boxed{\texttt{F}}$$

Indeed, this can be seen as follows. Let $M_{a_1,\dots,a_m}$ denote the number of inversions in a random word. If we distinguish the $a_i$ occurrences of $i$ by adding different fractional parts, in random order, the number of inversions will increase by $Z_i$, say, with the same distribution as $M_{a_i}$; further $M_{a_1,\dots,a_m}$ and $Z_1, \dots, Z_m$ are independent. On the other hand, $M_{a_1,\dots,a_m} + Z_1 + \cdots + Z_m$ has the same distribution as $M_{a_1+\cdots+a_m}$. Hence,

$$F_{a_1,\dots,a_m}(q) F_{a_1}(q) \cdots F_{a_m}(q) = F_{a_1+\cdots+a_m}(q), \qquad (1.7) \quad \boxed{\texttt{FF}}$$

which is (1.6).

By (1.6), we further have the factorization

$$F_{a_1,\ldots,a_m}(q) = \prod_{j=2}^{m} F_{A_{j-1},a_j}(q), \tag{1.8}$$ F2

where $A_j := a_1 + \cdots + a_j$, which reduces the general case to the two-letter case.

Note that (1.6) shows that the distribution of $M_{a_1,\ldots,a_m}$ is invariant if we permute $a_1, \ldots, a_m$; a symmetry which is not obvious from the definition.

Rpart **Remark 1.1.** The two-letter case is particularly interesting, since the unnormalized generating function

$$\binom{a+b}{a} F_{a,b}(q) = \frac{(1-q^{a+b})(1-q^{a+b-1})\cdots(1-q^{a+1})}{(1-q^{b})(1-q^{b-1})\cdots(1-q^{1})} = \frac{[a+b]!}{[a]!\,[b]!} \quad ,$$

(the $q$-binomial coefficient in (1.5)) is the same as the generating function for the set of integer-partitions with largest part $\leq a$ and $\leq b$ parts, in other words the set of integer-partitions whose Ferrers diagram lies inside an $a$ by $b$ rectangle, where the random variable is the "number of dots" (i.e. the integer being partitioned). In other words, the number of such partitions of an integer $n$ equal the number of words of $a$ 1's and $b$ 2's with $n$ inversions.

It is easy to see that the *mean* of $M_{a_1,\ldots,a_m}$ is

$$\mu(a_1,\ldots,a_m) := \mathbb{E}\, M_{a_1,\ldots,a_m} = e_2(a_1,\ldots,a_m)/2$$

(here $e_k(a_1,\ldots,a_m)$ is the degree $k$ elementary symmetric function), so considering the shifted random variable $M_{a_1,\ldots,a_m} - \mu(a_1,\ldots,a_m)$, "number of inversions minus the mean", we get that the probability generating function is

$$G_{a_1,\ldots,a_m}(q) := q^{-\mu(a_1,\ldots,a_m)} F_{a_1,\ldots,a_m}(q) = \frac{F_{a_1,\ldots,a_m}(q)}{q^{e_2(a_1,\ldots,a_k)/2}} \tag{1.9}$$ G

By computing $(q(qG)')'$ and plugging-in $q = 1$, or from (1.7) and (1.3)–(1.4), it is easy to see that the *variance* $\sigma^2 := \operatorname{Var} M_{a_1,\ldots,a_m}$ is

$$\sigma^2 = \frac{(e_1+1)e_2 - e_3}{12} \quad . \tag{1.10}$$ sigma

(By $\sigma$ we mean $\sigma(a_1,\ldots,a_m)$ and we omit the arguments $(a_1,\ldots,a_m)$ from the $e_i$'s.)

Let $N := e_1 = a_1 + \cdots + a_m$, the length of the random word, and let $a^* := \max_j a_j$ and $N_* := N - a^*$.

One main result of the present article is:

T1 **Theorem 1.2.** *Consider the random variable, $M_{a_1,\ldots,a_m}$, "number of inversions", on the (uniform) sample space of words with $a_1$ 1's, $a_2$ 2's, ..., $a_m$ $m$'s. For any sequence of sequences $(a_1,\ldots,a_m) = (a_1^{(\nu)},\ldots,a_{m^{(\nu)}}^{(\nu)})$ such* *that $N_* := N - a^* \to \infty$, the sequence of normalized random variables*

$$X_{a_1,\dots,a_m} = \frac{M_{a_1,\dots,a_m} - \mu(a_1,\dots,a_m)}{\sigma(a_1,\dots,a_m)} \quad ,$$

*tends to the standard normal distribution $\mathcal{N}(0,1)$, as $\nu \to \infty$.*

Theorem 1.2 includes both the case when $m \geq 2$ is fixed, and the case when $m \to \infty$. If $m$ is fixed and $a_1 \geq a_2 \geq \cdots \geq a_m$, as may be assumed by symmetry, then the condition $N_* \to \infty$ is equivalent to $a_2 \to \infty$. In the case $m \to \infty$, the assumption $N_* \to \infty$ is redundant, because $N_* \geq m - 1$.

**Remark 1.3.** The condition $N_* \to \infty$ is also necessary for asymptotic normality, see Section 6.

We give a short proof of this result using characteristic functions in Section 3. We give first in Section 2 another proof (at least of a special case) that is *computer-assisted*, using the Maple package `MahonianStat` available from the webpage of this article:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mahon.html`,
where one can also find sample input and output. This first proof uses the *method of moments.*

We conjecture that Theorem 1.2 can be refined to a local limit theorem as follows:

Conjlocal **Conjecture 1.4.** *Uniformly for all $a_1,\dots,a_m$ and all integers $k$,*

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma}\Bigl(e^{-(k-\mu)^2/(2\sigma^2)} + O\Bigl(\frac{1}{N_*}\Bigr)\Bigr). \qquad (1.11)$$ local

We have not been able to prove this conjecture in full generality, but we prove it under additional hypotheses on $a_1,\dots,a_m$ in Section 4.

Spf1

## 2. A computer-inspired proof

We assume for simplicity that $m$ is fixed, and that $(a_1,\dots,a_m) = (ta_1^0,\dots,ta_m^0)$ for some fixed $a_1^0,\dots,a_m^0$ and $t \to \infty$.

We discover and prove the leading term in the asymptotic expansion, in $t$, for an *arbitrary* $2r$-th moment, for the normalized random variable $X_{a_1,\dots,a_m} = (M_{a_1,\dots,a_m} - \mu)/\sigma$, and show that it converges to the moment $\mu_{2r} = (2r)!/(2^r r!)$ of $\mathcal{N}(0,1)$, for every $r$.

For the sake of exposition, we will only treat in detail the two-letter case, where we can find *explicit* expressions for the asymptotics of the $2r$-th moment, for symbolic $a_1, a_2, t$ and $r$ to *any* desired (specific) order $s$ (i.e. the leading coefficient $t^{3r}$ as well as the terms involving $t^{3r-1},\dots,t^{3r-s}$). A modified argument works for the general case, but we can only find the leading term, i.e. that

$$\alpha_{2r} := \mathbb{E}(X_{a_1,\dots,a_m})^{2r} = \frac{(2r)!}{2^r r!}t^{3r} + O(t^{3r-1}) \quad .$$

Of course the odd moments are all zero, since the distribution of $M_{a_1,\dots,a_m}$ is symmetric about $\mu$.

The mean in the two-letter case is simply $ab/2$, so the probability generating function for $M_{a,b}-\mu$ is

$$G_{a,b}(q)=\frac{F_{a,b}(q)}{q^{ab/2}(a+b)!/(a!b!)}=\frac{a!b!(1-q^{a+b})(1-q^{a+b-1})\cdots(1-q^{a+1})}{q^{ab/2}(a+b)!(1-q^{b})(1-q^{b-1})\cdots(1-q^{1})} \quad ,$$

Taking ratios, we have:

$$\frac{G_{a,b}(q)}{G_{a-1,b}(q)}=\frac{a(1-q^{a+b})}{q^{b/2}(a+b)(1-q^{a})} \quad . \tag{2.1}$$ ratio

Recall that the binomial moments $A_r := \mathbb{E}[\binom{X}{r}]$ are the Taylor coefficients of the probability generating function (in our case $G_{a,b}(q)$) around $q=1$. Writing $q=1+z$, we have

$$G_{a,b}(1+z)=\sum_{r=0}^{\infty} A_r(a,b)z^r$$

Note that $A_0(a,b)=1$ and $A_1(a,b)=0$. Let us call the expression on the right side of (2.1), with $q$ replaced by $1+z$, $P(a,b,z)$:

$$P(a,b,z):=\frac{a(1-(1+z)^{a+b})}{(1+z)^{b/2}(a+b)(1-(1+z)^{a})}$$

Maple can easily expand $P(a,b,z)$ to any desired power of $z$, It starts out with

$$P(a,b,z)=1+\frac{1}{24}\left(2\,a+b\right)bz^2-\frac{1}{24}\left(2\,a+b\right)bz^3 \\ -\frac{1}{5760}\left(8\,a^3-8\,a^2b-12\,ab^2-3\,b^3-440\,a-220\,b\right)bz^4+\ldots$$

note that the coefficients of all the powers of $z$ are polynomials in $(a,b)$.

So let us write

$$P(a,b,z)=\sum_{i=0}^{\infty} p_i(a,b)z^i \quad ,$$

where $p_i(a,b)$ are certain polynomials that Maple can compute for any $i$, no matter how big.

Looking at the recurrence

$$G_{a,b}(1+z)=P(a,b,z)G_{a-1,b}(1+z) \quad ,$$

and comparing coefficients of $z^r$ on both sides, we get

$$A_r(a,b)-A_r(a-1,b)=\sum_{s=1}^{r} A_{r-s}(a-1,b)p_s(a,b) \quad . \tag{2.2}$$ recurrence

Assuming that we already know the polynomials $A_{r-1}(a,b), A_{r-2}(a,b),\ldots,$ $A_0(a,b)$, the left side is a certain specific polynomial in $a$ and $b$, that Maple can easily compute, and then $A_r(a,b)$ is simply the indefinite sum of that polynomial, that Maple can do just as easily. So (2.2) enables us to get *explicit* expressions for the binomial moments $A_r(a,b)$ for *any* (numeric) $r$.

But what about the general (symbolic) $r$? It is too much to hope for the full expression, **but** we can easily conjecture as many leading terms as we wish. We first conjecture, and then immediately prove by induction, that for $r \geq 1$

$$A_{2r}(a,b) = \frac{1}{r!}\left(\frac{ab(a+b)}{24}\right)^r + \text{lower order terms}$$
$$A_{2r+1}(a,b) = \frac{-1}{(r-1)!}\left(\frac{ab(a+b)}{24}\right)^r + \text{lower order terms} \quad ,$$

where we can conjecture (by fitting polynomials in $(a,b)$ to the data obtained from the numerical $r$'s) any (finite, specific) number of terms.

Once we have asymptotics, to any desired order, for the binomial moments, we can easily compute the moments $\mu_r(a,b)$ of $M_{a,b} - \mu$ themselves, for *any* desired specific $r$ and asymptotically, to any desired order. We do that by using the expressions of the powers as linear combination of falling-factorials (or equivalently binomials) in terms of Stirling numbers of the second kind, $S(n,k)$. Note that for the asymptotic expressions to any desired order, we can still do it symbolically, since for any specific $m$, $S(n, n-m)$ is a polynomial in $n$ (that Maple can easily compute, symbolically, as a polynomial in $n$). In particular, the variance is:

$$\sigma^2 = \mu_2(a,b) = \frac{ab(a+b+1)}{12} \quad ,$$

in accordance with (1.10). In general we have $\mu_{2r+1}(a,b) = 0$, of course, and the six leading terms of $\mu_{2r}(at,bt)$ can be found in the webpage of this article. From this, Maple finds that $\alpha_{2r}(at,bt) := \mu_{2r}(at,bt)/\mu_2(at,bt)^r$ are given asymptotically (for fixed $a,b$ and $t \to \infty$) by:

$$\alpha_{2r}(at,bt) = \frac{(2r)!}{2^r r!} \cdot \left(1 - \frac{r(r-1)\left(b^2+ab+a^2\right)}{5ab\,(a+b)} \cdot \frac{1}{t}\right) + O(t^{-2}) \quad .$$

In particular, as $t \to \infty$, they converge to the famous moments of $\mathcal{N}(0,1)$. QED.

2.1. **The general case.** To merely prove asymptotic normality, one does not need a computer, since we only need the leading terms. The above proof can be easily adapted to the general case $(a_1, \ldots, a_m) = (ta_1^0, \ldots, ta_m^0)$. One simply uses induction on $m$, the number of different letters.

2.2. **The Maple package MahonianStat.** The Maple package `MahonianStat`, accompanying this article, has lots of features, that the readers can explore at their leisure. Once downloaded into a directory, one goes into a maple session, and types `read MahonianStat`. To get a list of the main procedures, type: `ezra();`. To get help with a specific procedure, type `ezra(ProcedureName);`. Let us just mention some of the more important procedures.

`AsyAlphaW2tS(r,a,b,t,s)`: inputs symbols `r,a,b,t` and a positive integer $s$, and outputs the asymptotic expansion, to order $s$, for $\alpha_{2r}$ $(=\mu_{2r}/\mu_2^r)$

`ithMomWktE(r,e,t)` the $r$-th moment about the mean of the number of inversions of $a_1t$ 1's, ..., $a_mt$ n's in terms of the elementary symmetric functions, in $a_1,\dots,a_m$. Here $r$ is a specific (numeric) positive integer, but $e$ and $t$ are symbolic.

`AppxWk(L,x)`: Using the asymptotics implied by the asymptotic normality of the (normalized) random variable under consideration, finds an approximate value for the number of words with $L[1]$ 1's, $L[2]$ 2's, ..., $L[n]$ $n$'s with exactly $x$ inversions. For example, try: `AppxWk([100,100,100],15000);`

For the two-lettered case, one can get better approximations, by procedure `BetterAppxW2`, that uses improved limit-distributions, using more terms in the probability density function.

The webpage of this article has some sample input and output.

## 3. A general proof of Theorem 1.2

Spf2

We have an exact formula (1.10) for the variance $\sigma^2$ of $M_{a_1,\dots,a_m}$. We first show that $\sigma$ 2 is always of the order $\Theta(N^2N_*)$.

Lsigma **Lemma 3.1.** *For any $a_1,\dots,a_m$,*

$$\frac{N^2N_*}{36}\le\sigma^2\le\frac{(N+1)NN_*}{12}\le\frac{N^2N_*}{6}.$$

*Proof.* For the upper bounds we assume, by symmetry, that $a_1\ge\cdots\ge a_m$. Then $a^*=a_1$ and

$$e_2=a_1\sum_{j=2}^m a_j+a_2\sum_{j=3}^m a_j+\cdots\le N\sum_{j=2}^m a_j=NN_*.$$

Since $e_1=N$, (1.10) yields the upper bounds.

For the lower bound, we first observe that $2e_2e_1-6e_3\ge 0$ (since this difference can be written as a sum of certain $a_ja_ka_l$). Hence $e_3\le e_1e_2/3$ and (1.10) yields

$$12\sigma^2\ge e_1e_2-e_3\ge\tfrac{2}{3}e_1e_2.$$

Further,

$$2e_2=\sum_{j=1}^m a_j(N-a_j)\ge\sum_{j=1}^m a_j(N-a^*)=NN_*,$$

and the lower bound follows. □

*Proof of Theorem 1.2.* From (1.6) follows the identity

$$F_{n_1,n_2}(e^{\mathrm{i}\theta})\;=\;\prod_{j=1}^{n_2}\frac{(e^{\mathrm{i}(n_1+j)\theta}-1)/(\mathrm{i}(n_1+j)\theta)}{(e^{\mathrm{i}j\theta}-1)/(\mathrm{i}j\theta)}.\tag{3.1}$$ f2q

By Taylor's series

$$\log\frac{e^z-1}{z}=z/2+z^2/24+O(z^4),\qquad |z|\le 1,$$

and we substitute this expansion into the identity (3.1) to conclude:

$$F_{n_1,n_2}(e^{\mathrm{i}\theta}) = \exp\left(\mathrm{i}n_1n_2\theta/2 - n_1n_2(n_1+n_2+1)\theta^2/24 + O(n_2n_1^4\theta^4)\right), \tag{3.2}$$ r3

uniformly for $n_1 \geq n_2 \geq 1$ and $|\theta| \leq (n_1+n_2)^{-1}$.

We use the factorization (1.8). By symmetry, we may assume $a_1 \geq a_2 \geq \cdots \geq a_m$, and then $A_{j-1} \geq a_{j-1} \geq a_j$ for each $j$. Thus (3.2) yields, uniformly for $q = e^{\mathrm{i}\theta}$ with $|\theta| \leq N^{-1}$,

$$\begin{aligned} F_{a_1,\dots,a_m}(q) &= \prod_{j=2}^{m} F_{A_{j-1},a_j}(q) \\ &= \exp\left(\sum_{j=2}^{m}\left(\mathrm{i}A_{j-1}a_j\theta/2 - A_{j-1}a_j(A_j+1)\theta^2/24 + O(a_jA_{j-1}^4\theta^4)\right)\right). \end{aligned}$$

Here, the sums of the coefficients of $\theta$ and $\theta^2$ are easily evaluated, but we do not have to do that since they have to equal $\mathrm{i}\mu$ and $-\sigma^2/2$, respectively. Further,

$$\sum_{j=2}^{m} A_{j-1}^4 a_j \leq N^4 \sum_{j=2}^{m} a_j = N^4 N_*. \tag{3.3}$$ pa1

Consequently, if $|\theta| \leq N^{-1}$,

$$F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) = \exp\bigl(\mathrm{i}\mu\theta - \sigma^2\theta^2/2 + O(N^4N_*\theta^4)\bigr) \tag{3.4}$$ pa2

and, by (1.9),

$$G_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) = \exp\bigl(-\sigma^2\theta^2/2 + O(N^4N_*\theta^4)\bigr). \tag{3.5}$$ pag

Let $\theta = t/\sigma$. For any fixed $t$, by Lemma 3.1,

$$|Nt/\sigma| = O\bigl({N_*}^{-1/2}\bigr) = o(1),$$

so $|\theta| \leq N^{-1}$ if $\nu$ is large enough. Hence, by (3.3)–(3.5) and Lemma 3.1,

$$G_{a_1,\dots,a_m}(e^{\mathrm{i}t/\sigma}) == \exp\left(-\frac{t^2}{2} + O\Bigl(\frac{N^4N_*t^4}{N^4{N_*}^2}\Bigr)\right) = \exp\bigl(-t^2/2 + o(1)\bigr),$$

and Theorem 1.2 follows by the continuity theorem [4, Theorem XV.3.2]. □

Slocal

## 4. The local limit theorem

"If one can prove a central limit theorem for a sequence $a_n(k)$ of numbers arising in enumeration, then one has a qualitative feel for their behavior. A local limit theorem is better because it provides asymptotic information about $a_n(k)$ ...," [2]. In this section we prove that the relation (1.11) holds uniformly over certain very general, albeit not unrestricted, sets of tuples $\mathbf{a} = (a_1, \dots, a_m)$. The exact statement is given below in Theorem 4.5.

As explained in Bender [2], there are two standard conditions for passage from a central to a local limit theorem: (1) if the sequence in question is unimodal, then one has a local limit theorem for $n$ in the set $\{|n-\mu| \geq \epsilon\sigma\}$, $\epsilon > 0$; (2) if the sequence in question is log-concave, then one has a local limit theorem for all $n$. Our sequence, the coefficients of the $q$-mutinomial, is in fact unimodal, as first shown by Schur [6] using invariant theory, and later by O'Hara [5] using combinatorics. Unfortunately, the ensuing local limit theorem fails to cover the most interesting coefficients, the largest ones, near the mean $\mu$. However, our polynomials are manifestly not log-concave as is seen by inspecting the first three coefficients (assuming $n_1, n_2 \geq 2$)

$$\binom{n_1+n_2}{n_1}_q = 1 + q + 2q^2 + \cdots .$$

The question arises might the coefficients be log-concave near the mean, and here is a small table of empirical values: ($c[j] = [q^j]\binom{2n}{n}_q$)

| $n$ | $(c[n^2/2-1])^2 - c[n^2/2] \times c[n^2/2-2]$ |
|---|---:|
| 2 | -1 |
| 4 | -7 |
| 6 | -165 |
| 8 | -1529 |
| 10 | 44160 |
| 12 | 7715737 |
| 14 | 905559058 |
| 16 | 101507214165 |
| 18 | 11955335854893 |
| 20 | 1501943866215277 |

Based on this scant evidence, we speculate that some sort of log-concavity theorem is true, but that its proper statement is complicated by describing the appropriate range of $\mathbf{a}$ and $j$. Thus, we use neither of the two standard methods mentioned above for proving our local limit theorem. (Later, we shall see that our theorem has implications for log-concavity.) Instead, we use another standard method, direct integration (Fourier inversion) of the characteristic function, or equivalently of the probability generating function $F(q)$ for $q = e^{i\theta}$ on the unit circle. This method uses the estimates for small $\theta$ derived in Section 3 together with estimates of $|F(e^{i\theta})|$ for larger $\theta$. We begin with one such estimate for rather small $\theta$.

LG1 **Lemma 4.1.** *There exists a constant $\tau > 0$ such that for any $a_1, \ldots, a_m$ and $|\theta| \leq \tau/N$,*

$$\left|F_{a_1,\ldots,a_m}(e^{i\theta})\right| = \left|G_{a_1,\ldots,a_m}(e^{i\theta})\right| \leq e^{-\sigma^2\theta^2/4}.$$

*Proof.* Suppose that $0 < |\theta| \leq \tau/N$. Then, using Lemma 3.1,

$$\frac{N^4 N_* \theta^4}{\sigma^2\theta^2} \leq \frac{N^2 N_* \tau^2}{\sigma^2} \leq 36\tau^2,$$

so if $\tau$ is chosen small enough, the error term $O(N^4 N_* \theta^4)$ in (3.4) and (3.5) is $\leq \sigma^2\theta^2/4$, and thus the result follows from (3.5). $\square$

We let in the sequel $\tau$ denote this constant. We may assume $0 < \tau \le 1$.

Lmagnus **Lemma 4.2.** *Uniformly, for all $a_1, \dots, a_m$ and all integers $k$,*

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(k-\mu)^2/(2\sigma^2)} + \int_{\tau/N}^{\pi} |F_{a_1,\dots,a_m}(e^{i\theta})| \, d\theta + O\Big(\frac{1}{\sigma N_*}\Big).$$

*Proof.* For any integer $k$,

$$\begin{aligned}
&\mathbb{P}(M_{a_1,\dots,a_m} = k) - \frac{1}{\sqrt{2\pi}\sigma} e^{-(k-\mu)^2/(2\sigma^2)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_{a_1,\dots,a_m}(e^{i\theta}) e^{-ik\theta}\, d\theta - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma^2\theta^2/2} e^{-i(k-\mu)\theta}\, d\theta \\
&= \frac{1}{2\pi}\int_{|\theta|\le\tau/N} \Big(G_{a_1,\dots,a_m}(e^{i\theta}) - e^{\sigma^2\theta^2/2}\Big) e^{-i(k-\mu)\theta}\, d\theta \\
&\qquad + \frac{1}{2\pi}\int_{\tau/N\le|\theta|\le\pi} F_{a_1,\dots,a_m}(e^{i\theta}) e^{-ik\theta}\, d\theta \\
&\qquad - \frac{1}{2\pi}\int_{|\theta|\ge\tau/N} e^{-\sigma^2\theta^2/2} e^{-i(k-\mu)\theta}\, d\theta \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For $|\theta| \le \tau/N$, $|G_{a_1,\dots,a_m}(e^{i\theta})| \le e^{-\sigma^2\theta^2/4}$ by Lemma 4.1, so (3.5) and the mean-value theorem applied to $z \mapsto e^z$ yield

$$\begin{aligned}
\big|G_{a_1,\dots,a_m}(e^{i\theta}) - e^{-\sigma^2\theta^2/2}\big| &\le O(N^4 N_* \theta^4) \max(|G_{a_1,\dots,a_m}(e^{i\theta})|, e^{-\sigma^2\theta^2/2}) \\
&= O\big(N^4 N_* \theta^4 e^{-\sigma^2\theta^2/4}\big).
\end{aligned}$$

Integrating, we find, using Lemma 4.1,

$$\begin{aligned}
|I_1| &\le \int_{|\theta|\le\tau/N} \Big|G_{a_1,\dots,a_m}(e^{i\theta}) - e^{-\sigma^2\theta^2/2}\Big| \, d\theta \\
&\le O\big(N^4 N_*\big) \int_{-\infty}^{\infty} \theta^4 e^{-\sigma^2\theta^2/4}\, d\theta = O\big(N^4 N_* \sigma^{-5}\big) = O\Big(\frac{1}{\sigma N_*}\Big).
\end{aligned}$$

Further, again using Lemma 4.1,

$$|I_3| \le \int_{\tau/N}^{\infty} e^{-\sigma^2\theta^2/2}\, d\theta \le 3\sigma^{-1} e^{-(\sigma\tau/N)^2/2} \le \frac{6}{\sigma(\sigma\tau/N)^2} = O\Big(\frac{1}{\sigma N_*}\Big).$$

Finally, $|I_2| \le \int_{\tau/N}^{\pi} |F_{a_1,\dots,a_m}(e^{i\theta})| \, d\theta$. ∎

In order to verify Conjecture 1.4, it thus suffices to show that the integral $\int_{\tau/N}^{\pi} |F_{a_1,\dots,a_m}(e^{i\theta})| \, d\theta$ in Lemma 4.2 is $O\Big(\frac{1}{\sigma N_*}\Big)$.

**Remark 4.3.** For example, an estimate

$$F_{a_1,\dots,a_m}(e^{i\theta}) = O\Big(\frac{1}{\sigma^3\theta^3}\Big), \qquad 0 < \theta \le \pi, \tag{4.1}$$ conj3

is sufficient for (1.11). We conjecture that this estimate (4.1) holds when $N_* \ge 6$, say. Note that it does not hold for very small $N_*$: taking $\theta = \pi$ we have, for even $n_1$, $F_{n_1,1}(-1) = 1/(n_1+1) = 1/N$, and the same holds for $F_{n_1,2}(-1)$.

Note further that even the weaker estimate

$$F_{a_1,\dots,a_m}(e^{i\theta}) = O\Big(\frac{1}{\sigma^2\theta^2}\Big), \qquad 0<\theta\le\pi, \tag{4.2}$$ conj2

would be enough to prove (1.11) with the weaker error term $O({N_*}^{-1/2})$.

We obtain a partial proof of Conjecture 1.4 using the following lemma.

LC1 **Lemma 4.4.** *For a given $\tau \in (0,1]$ there exists $c = c(\tau) > 0$ such that*

$$|F_{n_1,n_2}(e^{i\theta})| \ \le \ e^{-cn_2} \tag{4.3}$$ r2

*for $n_1 \ge n_2 \ge 1$ and $\tau/(n_1+n_2) \le |\theta| \le \pi$.*

*More generally, for any $a_1,\dots,a_m$ and $\tau/N \le |\theta| \le \pi$,*

$$|F_{a_1,\dots,a_m}(e^{i\theta})| \ \le \ e^{-cN_*}. \tag{4.4}$$ r2m

*Proof.* We prove first (4.3). For positive integer $n$ define

$$f_n(y,q) = \prod_{j=0}^{n}(1-yq^j)^{-1}.$$

For $0 \le R < 1$, we have (e.g. by Taylor expansions) $e^{2R} \le \frac{1+R}{1-R}$, and thus $e^{4R} \le \frac{(1+R)^2}{(1-R)^2} = 1 + \frac{4R}{(1-R)^2}$. Hence, by convexity, for any real $\zeta$,

$$e^{2R(1-\cos\zeta)} \le 1 + \frac{2R(1-\cos\zeta)}{(1-R)^2} = \frac{1+R^2-2R\cos\zeta}{(1-R)^2} = \frac{|1-Re^{i\zeta}|^2}{(1-R)^2},$$

and thus

$$\Big|(1-Re^{i\zeta})^{-1}\Big| \ \le \ (1-R)^{-1}\exp\left(R(1-\cos\zeta)\right).$$

Consequently, by a simple trigonometric identity, for any real $\phi$ and $\theta$,

$$\begin{aligned}\Big|f_{n_1}(Re^{i\phi},e^{i\theta})\Big| \ &\le \ (1-R)^{-n_1-1} \\ &\qquad \times \exp\bigg(-R\Big(n_1+1-\cos\big(\phi+\frac{n_1}{2}\theta\big)\frac{\sin(n_1+1)\theta/2}{\sin\theta/2}\Big)\bigg) \\ &\le \ (1-R)^{-n_1-1} \times \exp\bigg(R\Big(-n_1-1+\frac{\sin(n_1+1)\theta/2}{\sin\theta/2}\Big)\bigg).\end{aligned}$$

The function $g(\theta) = g_n(\theta) := \frac{\sin n(\theta/2)}{\sin(\theta/2)}$, where $n\ge 1$, is an even function of $\theta$; is decreasing for $0\le\theta\le\pi/n$, as can be verified by calculating $g'$; and satisfies $|g(\theta)| \le g(\pi/n)$ for $\pi/n \le |\theta| \le \pi$. Further, for $n\ge 2$ and $0 \le |\theta| \le \pi/n$,

$$\begin{aligned}g_n(\theta) &= 2\frac{\sin(n\theta/4)}{\sin(\theta/2)}\cos(n\theta/4) = 2g_{n/2}(\theta)\cos(n\theta/4) \le n\cos(n\theta/4) \\ &\le n\Big(1-\frac{n^2\theta^2}{40}\Big).\end{aligned}$$

Let $\theta_0 = \tau(n_1+n_2)^{-1} < \pi/(n_1+1)$. For $\theta_0 \le |\theta| \le \pi$ we thus have

$$|g_{n_1+1}(\theta)| \le g_{n_1+1}(\theta_0) \ \le \ n_1 + 1 - \frac{n_1^3\theta_0^2}{40};$$

whence, for $0 \le R < 1$, the estimate above yields

$$\left|f_{n_1}(Re^{\mathrm{i}\phi}, e^{\mathrm{i}\theta})\right| \ \le \ (1-R)^{-n_1-1}\exp\left(-Rn_1^3\theta_0^2/40\right). \tag{4.5}$$ tho

Combinatorially we know that $[y^\ell q^n]f_{n_1}(y,q)$ is the number of partitions of $n$ having at most $\ell$ parts no one of which exceeds $n_1$. As said in Remark 1.1, this equals $[q^n]\binom{n_1+\ell}{n_1}F_{n_1+\ell}(q)$. Hence, using Cauchy's integral formula, for any $R > 0$,

$$\binom{n_1+n_2}{n_1}F_{n_1,n_2}(q) = [y^{n_2}]f_{n_1}(y,q) = \frac{1}{2\pi\mathrm{i}}\int_{|y|=R} f_{n_1}(y,q)\frac{\mathrm{d}y}{y^{n_2+1}} \quad .$$

Consequently, (4.5) implies that for $\theta_0 \le |\theta| \le \pi$ and $0 < R < 1$,

$$\binom{n_1+n_2}{n_1}\left|F_{n_1,n_2}(q)\right| \le (1-R)^{-n_1-1}R^{-n_2}\exp\left(-Rn_1^3\theta_0^2/40\right).$$

Now choose $R = \rho := n_2/(n_1+n_2) \le 1/2$. By Stirling's formula,

$$\binom{n_1+n_2}{n_1} = \Omega\big(n_2^{-1/2}\big)\,(1-\rho)^{-n_1-1}\rho^{-n_2}$$

and thus, for $\theta_0 \le |\theta| \le \pi$,

$$\left|F_{n_1,n_2}(q)\right| \le O(n_2^{1/2})\exp\left(-\rho n_1^3\theta_0^2/40\right) = O(n_2^{1/2})\exp\left(-\Omega(n_2)\right).$$

This shows (4.3) for $n_2$ large enough, say $n_2 \ge n_0$. The case $n_2 \le n_0$ can be XXX

XXX
I hope this case is simple, and perhaps can be dismissed without giving details. If not, I think that we do not really need it.

To prove (4.4), we assume as we may that $a_1 \ge \cdots \ge a_m$ and use the factorization (1.8). Let $J$ be the first index such that $a_2 + \cdots + a_J \ge N_*/2$. For $j \ge J$, then $A_{j-1} + a_j = A_j \ge A_J \ge a_1 + N_*/2 \ge N/2$, and thus $A_k|\theta| \ge N|\theta|/2 \ge \tau/2$; hence (4.3) yields

$$\left|F_{A_{j-1},a_j}(e^{\mathrm{i}\theta})\right| \le e^{-c(\tau/2)a_j}.$$

We thus obtain from (1.8), since each $F_{n_1,n_2}$ is a probability generating function and thus is bounded by 1 on the unit circle,

$$|F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})| = \prod_{j=2}^m |F_{A_{j-1},a_j}(e^{\mathrm{i}\theta})| \le \prod_{j=J}^m e^{-c(\tau/2)a_j} \le e^{-c(\tau/2)N_*/2},$$

because $\sum_{j=J}^m a_j \ge N_*/2$. This proves (4.4) (redefining $c(\tau)$). □

Tlocal **Theorem 4.5.** *There exists a positive constant $c$ such that for every $C$, the following is true. Uniformly for all $a_1,\dots,a_m$ such that $a^* \le Ce^{cN_*}$ and all integers $k$,*

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma}\Big(e^{-(k-\mu)^2/(2\sigma^2)} + O\Big(\frac{1}{N_*}\Big)\Big). \tag{4.6}$$ localxxx

*Proof.* Let $c_1 = c(\tau)$ be the constant in Lemma 4.4. Then, Lemmas 4.2 and 4.4 yield

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma}\Big(e^{-(k-\mu)^2/(2\sigma^2)} + O\Big(\frac{1}{N_*} + \sigma e^{-c_1 N_*}\Big)\Big).$$

For any fixed $c < c_1$ we have, using Lemma 3.1, $\sigma N_* e^{-c_1 N_*} = O(Ne^{-cN_*})$ and thus

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma}\Big(e^{-(k-\mu)^2/(2\sigma^2)} + O\Big(\frac{1+Ne^{-cN_*}}{N_*}\Big)\Big).$$

The result follows, since $Ne^{-cN_*} = a^* e^{-cN_*} + N_* e^{-cN_*} = a^* e^{-cN_*} + O(1)$. □

`[Old stuff to be edited:]` XXX

The combinatorics of the Macmahon $q$ statistic appear not to have been greatly used, but in fact play an essential, easily overlooked, role. It seems that the Taylor estimate (3.2) is the key for a central limit theorem. As in Bender [2], some kind of smoothness condition is needed to proceed to a local limit theorem (1). Lemma 1, the large $\theta$ bound, imposes the required smoothness. The proof of Lemma 1 is made possible by the following combinatorial property:

$$[q^n]\binom{n_1+n_2}{n_1}_q = \sum_{\ell=1}^{n_2}[y^\ell]\prod_{j=1}^{n_1}(1-yq^j)^{-1}.$$

Thus, our Theorem 4.5 may be the $h(x) = (1-x)^{-1}$ case of a more general theorem about

$$[q^n]\sum_{\ell=1}^{n_2}[y^\ell]\prod_{j=1}^{n_1}h(yq^j).$$

## 5. Log-concavity

`[Old stuff to be edited:]` XXX

Let us review the proof of Theorem 4.5 with the intention of greater accuracy. The goal is to prove log-concavity in some range. For concreteness, let $\mathbf{a} = (n,n)$. Then $\sigma^2$ is of order $n^3$, and for sufficient accuracy we take the Taylor series (3) out to $O(\theta^{10})$. This improves (5) to:

$$\frac{1}{2\pi}\int_{\text{small}\,\theta} F(e^{\mathrm{i}\theta})e^{-\mathrm{i}j\theta}d\theta \;=\; \frac{F(1)}{\sigma\sqrt{2\pi}}\left(e^{-x^2/2}\left(1+\frac{p_4(x)}{\sigma^4}+\frac{p_6(x)}{\sigma^6}+\frac{p_8(x)}{\sigma^8}\right)+O(\frac{1}{n^4})\right).$$

Here, $p_4(x),\dots$ are certain polynomials and $x$ is determined by $j = \mu + x\sigma$. The sizes of the terms involving the polynomials are $1, 1/n, 1/n^2, 1/n^3$. The large $\theta$ contribution is negligible by comparison. One may now use the formula to calculate the difference $c_j^2 - c_{j_1}c_{j+1}$ to accuracy $O(1/n^4)$. Taking out the obvious common factors, it is found to equal $1/\sigma^2$. This gives:

TC2 **Theorem 5.1** (A log-concavity result)**.** *For each constant $C$ we have $n_0$ such that for $n \ge n_0$*

$$c_j^2 \ge c_{j-1}c_{j+1},$$

*for $|j-\mu| \le C\sigma$. Here,*

$$c_j = [q^j]\binom{2n}{n}_q.$$

We note that the "mysterious" numbers appearing in our earlier table are asymptotically

$$\frac{1}{2\pi\sigma^4}\binom{2n}{n}^2.$$

Sfine

## 6. Final comments

Suppose that $N_* \not\to \infty$. We may, as usual, assume that $a_1 \geq \cdots \geq a_m$. By considering a subsequence (if necessary), we may assume that $N_* := N - a^* = a_2 + \cdots + a_m$ is a constant; this entails that $m$ is bounded, so by again considering a subsequence, we may assume that $m$ and $a_2, \dots, a_m$ are constant. We thus study the case when $a_1 \to \infty$ with fixed $a_2, \dots, a_m$.

In this case, the number of inversions between indices $2, \dots, m$ is $O(1)$, which is asymptotically negligible. Ignoring these, we can thus consider the random word as $N_*$ letters $2, \dots, m$ inserted in $a_1$ 1's, and the number of inversions is the sum of their positions, counted from the end. It follows easily, either probabilistically or by calculating the characteristic function from (1.6), that $M_{a_1,\dots,a_m}/N$, or equivalently $M_{a_1,\dots,a_m}/a_1$, converges in distribution to the sum $\sum_{j=1}^{N_*} U_j$ of $N_*$ independent random variables $U_j$ with the uniform distribution on $[0,1]$. Equivalently, since $\sigma^2 \sim n_1^2 N_*/12 \sim N^2 N_*/12$,

$$\frac{M_{a_1,\dots,a_m} - \mu(a_1,\dots,a_m)}{\sigma(a_1,\dots,a_m)} \xrightarrow{\mathrm{d}} \sqrt{\frac{12}{N_*}} \sum_{j=1}^{N_*} (U_j - \tfrac{1}{2}),$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution. This limit is clearly not normal for any finite $N_*$. (However, its distribution is close to standard normal for large $N_*$. Note that it is normalized to mean 0 and variance 1.)

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