The number of m-Dimensional Partitions of Eleven and Twelve

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Way back in the sixties, when computation was slow and difficult, in a true *tour de force*, Oliver Atkin (of *Ramanujan congruences* fame), Ian Macdonald (of *Symmetric Functions* and *Macdonald Polynomials* fame), John McKay (of *moonshine* fame), together with the *not-as-famous-but-nevertheless-computer-whiz* Paul Bratley, disproved an unfortunate conjecture of Major Percy Alexander MacMahon of *Combinatory Analysis* fame. To that end they derived explicit expressions, as polynomials in m, for $M_m(n)$, the number of m-dimensional partitions of n for $1 \leq n \leq 10$. Their work was beautifully, and leisurely, described by *Partitions guru* George Andrews ([A], ch. 11.). $M_m(n)$ agreed with MacMahon's conjecture (see [A]) for $n \leq 5$, so $M_m(6)$ sufficed to prove MacMahon wrong, and that was as far as Andrews went. But now that computers are so much faster, I asked my master, Doron Zeilberger, to spend an afternoon writing Maple code [Z] in order to extend these formulas to $n = 11$ and $n = 12$. He followed the strategy of [ABMM], but automating everything. Zeilberger didn't try too hard to "optimize", and using his suboptimal code, I easily got expressions for $M_m(n)$ for all $n \leq 12$. I am sure that if he tried a bit harder, he could have written a program that would get also $n = 13, 14$ and perhaps even $n = 15$, but who cares? I am willing to bet that no one in our lifetime would be able to compute $M_m(300)$. In fact, the Riemann Hypothesis would be proven way before the exact formula for $M_m(300)$ (a certain mundane polynomial of m of degree 299) would be ever known.

But since computing $M_m(n)$ for $n \leq 12$ was so easy, I am hereby filling a much needed gap and am stating the following extension of Theorem 11.8 of the classic [A]. For the sake of completeness I am also reproducing the previously known expressions for $n \leq 10$.

Theorem: Let $M_m(n)$ be the number of m-dimensional partitions of n, then

$$M_m(1) = 1 \quad ,$$

$$M_m(2) = m + 1 \quad ,$$

$$M_m(3) = 1/2\ (m+2)(m+1) \quad ,$$

$$M_m(4) = 1/6\ (m+1)\left(m^2 + 8\,m + 6\right) \quad ,$$

$$M_m(5) = 1/24\ (m+1)\left(m^3 + 21\,m^2 + 38\,m + 24\right) \quad ,$$

$$M_m(6) = \frac{1}{120}\ (m+4)(m+1)\left(m^3 + 40\,m^2 + 61\,m + 30\right) \quad ,$$

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Accompanied by Maple package http://www.math.rutgers.edu/~zeilberg/tokhniot/mDimPars written by D. Zeilberger .

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$$M_m(7) = \frac{1}{720}\,(m+1)\left(m^5+80\,m^4+965\,m^3+1810\,m^2+1824\,m+720\right) \quad ,$$

$$M_m(8) = \frac{1}{5040}\,(m+1)\left(m^6+132\,m^5+3235\,m^4+13320\,m^3+16864\,m^2+16848\,m+5040\right) \quad ,$$

$$M_m(9) = \frac{1}{40320}\,(m+1)\cdot$$
$$\left(m^7+203\,m^6+8911\,m^5+82145\,m^4+125944\,m^3+217532\,m^2+129744\,m+40320\right) \quad ,$$

$$M_m(10) = \frac{1}{362880}\,(m+1)\cdot$$
$$\left(m^8+296\,m^7+21238\,m^6+387884\,m^5+1273069\,m^4+1377404\,m^3+3261852\,m^2+935856\,m+362880\right) \quad ,$$

$$M_m(11) = \frac{1}{3628800}\,(m+1)\cdot$$
$$(m^9+414\,m^8+45366\,m^7+1464624\,m^6+10983189\,m^5+8292186\,m^4$$
$$+43268084\,m^3+18819576\,m^2+15104160\,m+3628800) \quad ,$$

$$M_m(12) = \frac{1}{39916800}\,(m+1)\cdot$$
$$(m^{10}+560\,m^9+88980\,m^8+4653120\,m^7+69709773\,m^6+152328120\,m^5$$
$$+98041670\,m^4+1093797880\,m^3-395547624\,m^2+473807520\,m+39916800) \quad .$$

Proof: Routine! (See the source code of [Z] for details).

Let me conclude by commenting that recently a renewed interest in the counting of higher-dimensional partitions (in physics!) has lead to the fascinating article [BGP]. It is possible that my "new" results are *implicit* there (recall that in order to decipher a polynomial of degree d it suffices to know $d+1$ data points, so in order to know, e.g., $M_m(13)$, we only need to know $M_m(13)$ for $1 \leq m \leq 13$, and this data may have already been computed by them (and if not, it could easily have been!))

References

[A] G. E. Andrews, *"The Theory of Partitions"*, Addison-Wesley, 1976. Reprinted by Cambridge University Press, 1984. First paperback edition, 1998.

[ABMM] A.O.L. Atkin, P. Bratley, I.G. Macdonald, and J.K.S. McKay, *Some Computations for m-dimensional partitions*, Proc. Cambridge Phil. Soc. **63** (1967), 1097-1100.

[BGP] S. Balakrishnan, S. Govindarajan and N. S. Prabhakar, *On the asymptotics of higher-dimensional partitions*, arXiv:1105.6231 .

[Z] D. Zeilberger, `mDimPars`, a Maple package available from
`http://www.math.rutgers.edu/~zeilberg/tokhniot/mDimPars` .