A One-Line Proof of Leversha's "Quartet of Isogonal Conjugates" Theorem

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In [L], Gerry Leversha spent quite a few pages, using human ingenuity and lots of previous knowledge, to prove the following elegant theorem (see [W] for the definition of *Isogonal Conjugate*).

Leversha's Quartet Theorem ([L]): Let $ABCD$ be a quadrilateral which is not cyclic. Let A^* be the isogonal conjugate of A with respect to ΔBCD, and similarly let B^*, C^*, D^* the isogonal conjugates of ΔADC, ΔABD, ΔABC respectively. Then A,B,C, and D are the circumcenters of $\Delta B^*C^*D^*$, $\Delta A^*C^*D^*$, $\Delta A^*D^*D^*$, and $\Delta A^*C^*B^*$.

Proof: Download the Maple package `RENE.txt` freely available from Doron Zeilberger's website `https://sites.math.rutgers.edu/~zeilberg/PG/RENE.txt` ,
start `Maple`, and type `read ‘RENE.txt‘:` followed by `Leversha();`, and in **one nano-second**, you would get that it is `true`. The Maple code for proving Leversha's theorem is as follows:

```
T1 := Te(m, n); T2 := Te(m1, n1); A := [0, 0]; B := T1[3]; C := T2[3];
evalb(DeSq(IsogonalConjugate(m1, n1, B), A) = DeSq(IsogonalConjugate(m, n, C), A));
```

Explanation

Without loss of generality (by translating, rotating, and shrinking) we can assume that $A = (0, 0)$, $D = (1, 0)$,

$$tan(\frac{1}{2}\angle BAD) = m \quad , \quad tan(\frac{1}{2}\angle ADB) = n \quad , \quad tan(\frac{1}{2}\angle CAD) = M \quad , \quad tan(\frac{1}{2}\angle ADC) = N \quad ,$$

so now ΔADB and ΔADC are what are called (in `RENE.txt`), `Te(m,n)` and `Te(M,N)`. Using the straightforward `IsogonalConjugate` *macro* we get B^* and C^* and we just check that the 'distance-squared' (`DeSq`), between A and B^* and A and C^* are the same. Of course, this is also true about the distance between A and B^* and A and D^*, and by the *transitivity* of the $=$ relation, all three distances are equal and hence indeed A is the circumcenter of $\Delta B^*C^*D^*$.

But using this 'analytic' approach (rather than Leversha's *synthetic* approach) gives much more. `LevershaRadius(m,n,M,N)` gives that the **exact** value of the circumradius of $\Delta B^*C^*D^*$ is

$$\frac{(N-n)\,(Nn+1)\left(m^2+1\right)\left(M^2+1\right)}{((MN-1)(mn-1)+(M+N)(n+m))\,(MN(n+m)-mn(M+N)+M+N-m-n)} \quad .$$

`RENE.txt` can also tell you the exact locations of B^* and C^*:

$$B^* = \Big(\frac{(Mm+M-m+1)\,(Mm-M+m+1)\,(Nn+1)\,(N-n)}{(MN(mn-1)+(M+N)(m+n)-mn+1)\,(MN(m+n)-mn(M+N)+M+N-m-n)} \, ,$$

$$\frac{2\,(Nn+1)\,(N-n)\,(Mm+1)\,(M-m)}{(MN(mn-1)+(M+N)(m+n)-mn+1)\,(MN(m+n)-mn(M+N)+M+N-m-n)} \Big) \quad ,$$

$$C^* = \left(\frac{(Mm+M-m+1)\,(Mm-M+m+1)\,(Nn+1)\,(N-n)}{(MN(mn-1)+(M+N)(m+n)-mn+1)\,(MN(m+n)-mn(M+N)+M+N-m-n)} \right.,$$

$$\left. -\frac{2\,(Nn+1)\,(N-n)\,(Mm+1)\,(M-m)}{(MN(mn-1)+(M+N)(m+n)-mn+1)\,(MN(m+n)-mn(M+N)+M+N-m-n)} \right) \quad .$$

References

[L] Gerry Leversha, *A quartet of isogonal conjugates*, The Mathematical Gazette, **100**, issue 548, (July, 2016), 336-341, (Note 100.23). Available from JSTOR.

[W] The Wikipedia Foundation, *Isogonal Conjugate*
`https://en.wikipedia.org/wiki/Isogonal_conjugate`

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