The Discrete Analog of the Malgrange-Ehrenpreis Theorem

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In fond meomory of Leon Ehrenpreis

One of the **landmarks** of the modern theory of partial **differential** equations is the **Malgrange-Ehrenpreis**[E][M] theorem (see [Wi]) that states that every non-zero linear partial differential operator with constant coefficients has a Green's function (alias **fundamental solution**). Recently Wagner[W] gave an elegant *constructive* proof.

In this short note I will state the **discrete** analog, and give two proofs. The first one is Ehrenpreis-style, using *duality*, and the second one is *constructive*, using *formal Laurent series*.

Let Z be the set of integers, and n a positive integer. Consider functions $f(m_1, \dots, m_n)$ from Z^n to the complex numbers (or any field). A **linear partial difference operator with constant coefficients** $\mathcal{P}$ is anything of the form

$$\mathcal{P} f(m_1, \dots, m_n) := \sum_{\alpha \in A} c_\alpha f(m_1 + \alpha_1, \dots, m_n + \alpha_n) \quad ,$$

where A is a **finite** subset of Z^n and $\alpha = (\alpha_1, \dots, \alpha_n)$, and the c_α are **constants**.

For example, the **discrete Laplace operator** in two dimensions:

$$f(m_1, m_2) \to f(m_1, m_2) - \frac{1}{4}(f(m_1+1, m_2) + f(m_1-1, m_2) + f(m_1, m_2+1) + f(m_1, m_2-1)) \quad .$$

The **symbol** of the operator $\mathcal{P}$ is the **Laurent polynomial**

$$P(z_1, \dots, z_n) = \sum_{\alpha \in A} c_\alpha z_1^{\alpha_1} \cdots, z_n^{\alpha_n} \quad .$$

The **discrete delta function** is defined in the obvious way

$$\delta(m_1, \dots, m_n) = \begin{cases} 1, & \text{if } (m_1, \dots, m_n) = (0, 0, \dots, 0); \\ 0, & \text{otherwise.} \end{cases}$$

Note that the beauty of the discrete world is that the delta function is a *genuine* function, not a "generalized" one, and one does not need the intimidating edifice of Schwartzian *distributions*

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`http://www.math.rutgers.edu/~zeilberg/` . First version: July 21, 2011. Accompanied by Maple package `LEON` available from `http://www.math.rutgers.edu/~zeilberg/tokhniot/LEON` . Supported in part by the USA National Science Foundation.

We are now ready to state the

Discrete Malgrange-Ehrenpreis Theorem: Let $\mathcal{P}$ be any non-zero **linear partial difference operator with constant coefficients**. There exists a function $f(m_1, \dots, m_n)$ defined on Z^n such that

$$\mathcal{P}f(m_1, \dots, m_n) = \delta(m_1, \dots, m_n) \quad .$$

First Proof (Ehrenpreis-style) Consider the infinite-dimensional vector space, $C[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]$, of *all* Laurent polynomials in $z_1, \dots, z_n$. Every function f on Z^n uniquely defines a **linear functional** T_f defined on monomials by

$$T_f(z_1^{m_1} \cdots z_n^{m_n}) := f(m_1, \dots, m_n) \quad ,$$

and extended by linearity. Conversely, any linear functional gives rise to a function on Z^n. Let $P(z_1, \dots, z_n)$ be the symbol of the operator $\mathcal{P}$. We are looking for a linear functional T such that for every monomial $z_1^{m_1} \cdots z_n^{m_n}$

$$T(P(z_1, \dots z_m) z_1^{m_1} \cdots z_n^{m_n}) = T_\delta(z_1^{m_1} \cdots z_n^{m_n}) \quad .$$

By linearlity, for *any* Laurent polynomial $a(z_1, \dots, z_n)$

$$T(P(z_1, \dots, z_n) a(z_1, \dots, z_n) = T_\delta(a(z_1, \dots, z_n)) \quad .$$

So T is defined on the (vector) **subspace** $P(z_1, \dots, z_n) C[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]$ of $C[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]$. By elementary linear algebra, every linear functional on the former can be **extended** (in many ways!) to the latter. **QED**.

Before embarking on the second proof we have to recall the notion of **formal power series** and more generally **formal Laurent series**.

A *formal power series* in one variable z is any creature of the form

$$\sum_{i=0}^{\infty} a_i z^i \quad .$$

More generally, a **positive** *formal Laurent series* is any creature of the form

$$\sum_{i \geq m}^{\infty} a_i z^i \quad ,$$

where m is a (possibly negative) integer. On the other hand a **negative** *formal Laurent series* is any creature of the form

$$\sum_{i \leq m}^{\infty} a_i z^i \quad ,$$

where m is a (possibly positive) integer.

A **bilateral** formal Laurent series goes **both ways**

$$\sum_{i=-\infty}^{\infty} a_i z^i \quad .$$

Note that the class of bilateral formal Laurent series is an abelian additive group, but one **can't** multiply there. On the other hand one can legally multiply two positive formal Laurent series by each other, and two negative formal Laurent series by each other, but don't mix them! Of course it is always legal to multiply any Laurent polynomial by any bilateral formal power series. **But watch out for** *zero-divisors*, e.g.

$$(1-z)\sum_{i=-\infty}^{\infty} z^i = 0 \quad .$$

Any Laurent polynomial $p(z) = a_i z^i + \ldots a_j z^j$ of low-degree i and (high-)degree j in z (so $a_i \neq 0$, $a_j \neq 0$) has **two** multiplicative inverses. One in the ring of positive Laurent polynomials, and the other in the ring of negative Laurent polynomials. Simply write $p(z) = z^i a_i p_0(z)$ and get $1/p(z) = z^{-i}(1/a_i)p_0(z)^{-1}$, and writing $p_0(z) = 1 + q_0(z)$, we form

$$p_0(z)^{-1} = (1+q_0(z))^{-1} = \sum_{i=0}^{\infty} (-1)^i q_0(z)^i \quad ,$$

and this makes perfect sense and *converges* in the ring of formal power series. Analogously one can form a multiplicative inverse in powers in z^{-1}.

It follows that every *rational function* $P(z)/Q(z)$ in one variable, z, has (at least) two inverses, one pointing positively, one negatively.

What about a rational function of several variables, $P(z_1, \ldots, z_n)/Q(z_1, \ldots, z_n)$? Here we can form $2^n n!$ inverses. There are $n!$ ways to order the variables, and for each of these one can decide whether to do the postive-pointing inverse or the negative-pointing one. At each stage we get a formal one-sided formal Laurent series whose coefficients are rational functions of the remaining variables, and one just keeps going.

Second Proof (Constructive): To every discrete function $f(m_1, \ldots, m_n)$ associate the bilateral formal Laurent series

$$\sum_{(m_1,\ldots,m_n)\in Z^n} f(m_1, \ldots, m_n) z_1^{m_1} \cdots z_n^{m_n} \quad .$$

We need to "solve" The equation

$$P(z_1^{-1}, \ldots, z_n^{-1}) \left(\sum_{(m_1,\ldots,m_n)\in Z^n} f(m_1, \ldots, m_n) z_1^{m_1} \cdots z_n^{m_n} \right) = 1 \quad .$$

So "explicitly"

$$\sum_{(m_1,\ldots,m_n)\in Z^n} f(m_1,\ldots,m_n) z_1^{m_1}\cdots z_n^{m_n} = 1/P(z_1^{-1},\ldots,z_n^{-1}) \quad ,$$

and we just described how to do it in $2^n n!$ ways.

The Maple package LEON

This article is accompanied by a Maple package `LEON`. One of its numerous procedures is `FS`, that implements the above constructive proof. `LEON` can also compute **polynomial bases** to solutions of partial difference equations, compute *Hilbert Series* for spaces of solutions of systems of linear differential equations, as well as find *multiplicity varieties*, in the style of Ehrenpreis, to 0-dimensional ones.

Leon Ehrenpreis (1930-2010) A truly FUNDAMENTAL Mathematician (a Videotaped lecture)

I strongly urge readers to watch my lecture, available in six parts from `YouTube`, and in two parts from `Vimeo`, see:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/leon.html` .

That page contains links to both versions, as well as numerous input and output files for the Maple package `LEON`.

References

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