# Lehmer's Conjecture on the Non-vanishing of Ramanujan's Tau Function

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**Abstract**

In this paper we prove Lehmer's conjecture on Ramanujan's tau function, namely $\tau(n) \neq 0$ for each $n \geq 1$ by investigating the additive group structure attached to $\tau(n)$ with the aid of unique factorization theorem.

**Note added Sept. 11, 2014 by the uploader, Doron Zeilberger**: I was told that the present proof is incomplete. Will Lee informed me that he is currently working on fixing it.

[1] Let $E_k$ $(k = 2, 4, \dots)$ be the normalized Eisenstein series ($[4 : 108 - 122]$) given by

$$E_k = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \tag{1}$$

where $q := e^{i2\pi z}$ $(\Im(z) > 0)$, $B_k$ the Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

and $\sigma_{k-1}(n)$ the divisor function:

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

For an elliptic curve given by

$$y^2 = 4x^3 - g_2(z)x - g_3(z) \tag{2}$$

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where $g_2(z) = 120\zeta(4)E_4(z), g_3(z) = 280\zeta(6)E_6(z)$ and $E_k(z)$ given by equation (1) and $\zeta(k)$ is Riemann zeta function:

$$\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

A simple calculation ([1 : 14], [4 : 112]) shows the discriminant $\varDelta(z) := 4^4(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$, where $x_1, x_2$ and $x_3$ are the roots the right side of equation (2), is given by

$$\varDelta(z) = g_2(z)^3 - 27g_3(z)^2 = \frac{(2\pi)^{12}}{1728}(E_4(z)^3 - E_6(z)^2). \tag{3}$$

On the other hand Jacobi's theorem ([4 : 122]) asserts that

$$(2\pi)^{-12}\varDelta(z) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}. \tag{4}$$

From equation (4), Ramanujan has defined his tau function $\tau(n)$ ([1], [2], [3], [4 : 122], [5] – [7]) by

$$q\prod_{n=1}^{\infty}(1 - q^n)^{24} := \sum_{n=1}^{\infty}\tau(n)q^n. \tag{5}$$

Notice that each $\tau(n)$ $(n \geq 1)$ has an integer value. In a series of papers ([5] – [7]), D.H. Lehmer investigated the properties of $\tau(n)$ for $n \leq 300$, proved that $\tau(n) \neq 0$ for $n < 3316799$, later for $n < 214928639999$ ([1 : 22]). He also showed that if $\tau(n) = 0$ then $n$ must be a prime. He then conjectured, what is nowadays known as Lehmer's conjecture ([6]) that

$$\tau(n) \neq 0 \quad \text{for} \quad \text{each} \quad n \geq 1. \tag{6}$$

A simple calculation ([3 : 21 – 22], [4 : 122 – 123]) shows

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{j=1}^{n-1}\sigma_5(j)\sigma_5(n-j). \tag{7}$$

Since Lehmer's conjecture is equivalent to $3\tau(n) \neq 0$ for each $n \geq 1$, we write

$$A(n) := \frac{65}{252}\sigma_{11}(n) + \frac{691}{252}\sigma_5(n);\ B(n) := 691\sum_{j=1}^{n-1} \sigma_5(j)\sigma_5(n-j). \tag{8}$$

Then $3\tau(n) = A(n) - B(n)$. Observe that $A(n)$ takes on integer value for each $n \geq 1$ since both $\tau(n)$ and $B(n)$ do. Now Lehmer's conjecture is, in view of equations (7), (8) and the unique factorization theorem, equivalent to:

$$A(n) \neq B(n) \ \text{ for each } \ n \geq 1. \tag{9}$$

Recent calculation by Bosman confirms Lehmer's conjecture for $n \leq 22798241520242687999$. In this paper we prove equation (9) by showing that $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{k=0}^{q-1}$ forms an additive group of order $q$ modulo $q$ for $q \mid A(p),\ q > p,\ p \equiv -1 \bmod 691,\ [a_{i,k}]_{0\leq i,k\leq q-1}$ $q \times q$-matrix, with the aid of the unique factorization theorem, the pigeonhole principle and the remainder theorem. We prove equation (9) first for prime $p$ then for $p^\alpha, \alpha \geq 2$ and finally for any composite number $n$. Since $11 \nmid 690$ and since $(p+1) \mid (p^{11}+1)$, the following Lemma 1 evidently holds.

**Lemma 1** *Let* $A(p)$ *be given by equation* (8). *Then the following two conditions* (10) *and* (11) *are equivalent:*

$$A(p) \equiv 0 \mod 691. \tag{10}$$

$$p \equiv -1 \mod 691. \tag{11}$$

If $691 \nmid A(p)$ or equivalently $p$ does not satisfy equation (11) then we trivially have $A(p) \neq B(p)$ by equation (8). It suffices therefore to prove Lehmer's conjecture for prime $p$ satisfying equation (10) or (11). In what follows, prime $p$ satisfies either equation (10) or (11). We first prove:

**Lemma 2** *Let $p$ satisfy equation* (10) *or* (11). *Then $A(p)$ has at least one prime factor $q$ greater than $p$.*

**Proof.** Write $A(p)$ from equation (8) as

$$\begin{aligned} A(p) &= \tfrac{65}{252}(1+p^{11}) + \tfrac{691}{252}(1+p^5) \\ &= 3 + K_5 p^5 \end{aligned} \tag{12}$$

where $K_5 := \frac{691}{252} + \frac{65}{252}p^6$ is an integer with $p^5 < K_5 < p^6$ since $A(p)$ is an integer with $p^{10} < A(p) < p^{11}$. Suppose $A(p)$ has no prime factor greater than $p$. $A(p)$ then is written via the unique factorization theorem as

$$A(p) = 2^{e_0} q_1^{e_1} \dots q_m^{e_m},\ q_i < p,\ e_i \geq 1\ (1 \leq i \leq m). \tag{13}$$

Notice that A(p) has an even factor $2^{e_0}$ by substituting equation (11) into equation (8). Write $x = [x] + \{x\}$ where $[x]$ represents the integral part of $x$ and $\{x\}$ the nonintegral part of $x$. Since $p^{10} < A(p) < p^{11}$, the representation for $A(p)$ in the base $p$ is uniquely given from equation (13) by

$$A(p) = \sum_{i=0}^{10} b_i p^i,\ b_i := [p\{\frac{A(p)}{p^{i+1}}\}]\ (0 \leq i \leq 10). \tag{14}$$

We show that each $b_i \neq 0$ $(0 \leq i \leq 10)$. Indeed we prove

$$b_i \geq 5\ (1 \leq i \leq 10),\ b_0 \neq 0. \tag{15}$$

The fact that $b_0 \neq 0$ follows from $p \nmid A(p)$. Likewise $b_{10} \geq 5$ follows from $p \nmid A(p)$ and $b_{10} \geq [\frac{65}{252}p] \geq [\frac{p}{252}] \geq 5$ in view of equations (14) and (17), where $p \geq 1381$. ¿From equation (8) or (12), we readily have

$$\frac{65}{252}p^{11} < A(p) < (\frac{65}{252} + 3p^{-6})p^{11}. \tag{16}$$

Equation (16) is equivalent to

$$\frac{65}{252}p^{10-i} < \frac{A(p)}{p^{i+1}} < (\frac{65}{252} + 3p^{-6})p^{10-i} \ (1 \le i \le 9). \tag{17}$$

Write

$$65p^{10-i} = 252Q_i + R_i, \ Q_i \ge 1, \ 1 \le R_i \le 251 \ (1 \le i \le 9). \tag{18}$$

Substitution of equation (18) into equation (17) reveals

$$Q_i + \frac{R_i}{252} < \frac{A(p)}{p^{i+1}} < (Q_i + \frac{R_i}{252} + 3p^{4-i}) \ (1 \le i \le 9). \tag{19}$$

Inequality (19) implies $\{\frac{A(p)}{p^{i+1}}\} \ge \frac{R_i}{252} \ge \frac{1}{252}$ $(1 \le i \le 9)$ and hence we have since $p \ge 1381$

$$b_i = [p\{\frac{A(p)}{p^{i+1}}\}] \ge [\frac{p}{252}] \ge 5 \ (1 \le i \le 9). \tag{20}$$

This establishes inequality (15). Rewrite equation (14) as

$$A(p) = L_5 p^5 + \sum_{i=0}^{4} b_i p^i \tag{21}$$

where $L_5 := \sum_{i=0}^{5} b_{5+i}p^i$. Subtraction of equation (21) from equation (12) with rearrangement of terms leads us to

$$\begin{array}{ll} p^5 & \le (K_5 - L_5)p^5 \\ & = (b_0 - 3) + \sum_{i=1}^{4} b_i p^i \\ & < (p-1)\sum_{i=0}^{4} p^i \\ & = p^5 - 1. \end{array} \tag{22}$$

Since each $b_i \ge 1$ $(0 \le i \le 10)$ from equation (15), $K_5 - L_5 \ge 1$ follows from the the second line equality of equation (22) regardless of the value of $b_0 \ne 0$. Since $1 \le b_i \le p-1$ $(0 \le i \le 10)$ from equation (15) and since $b_0 - 3 < p - 1$, the third line inequality follows. Inequality (22) is absurd. Consequently the assumption that $A(p)$ has no prime factor $> p$ is false. This establishes Lemma 2.

It is easy to check our proof for Lemma 2 works for all primes $p > 252$ with inequality (15) replaced by $1 \leq b_i$ $(0 \leq i \leq 10)$. A simple computation reveals Lemma 2 also holds for primes $p \leq 252$. Consequently Lemma 2 holds for all primes $p$. Thus the assumption that the prime $p$ generated by equation (11) in Lemma 2 is redundant.

Let $q$ be an odd prime prime factor of $A(p)$ greater than p. Existence of such a prime $q$ is guaranteed by Lemma 2. Construct matrix $[a_{i,k}]_{0\leq i,k\leq q-1}$ as follows:

$$a_{i,k} := \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \bmod q}}^{p-1} 1. \tag{23}$$

Since $\sigma_5(j)\sigma_5(p-j) = \sigma_5(p-j)\sigma_5(p-(p-j))$, we have from equation (23)

$$a_{i,k} = 2 \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \bmod q}}^{(p-1)/2} 1. \tag{24}$$

Then the matrix $[a_{i,k}]_{0\leq i,\, k\leq q-1}$ has the following properties:

$$\begin{aligned}
a_{i,k} &\equiv 0 \mod 2 && (0 \leq i,k \leq q-1). && (25)\\
a_{0,0} &= p-1,\ a_{0,k} = 0 && (1 \leq k \leq q-1). && (26)\\
a_{i,0} &= a_{j,0} && (1 \leq i \neq j \leq q-1). && (27)\\
a_{i,k} &= a_{q-i,q-k} && (1 \leq i,k \leq q-1). && (28)\\
i691\sum_{j=1}^{p-1}\sigma_5(j)\sigma_5(p-j) &\equiv \sum_{k=1}^{q-1} k a_{i,k} \mod q && (1 \leq i \leq q-1). && (29)\\
\sum_{k=1}^{q-1} k a_{i,k} &\equiv i\sum_{k=1}^{q-1} k a_{1,k} \mod q && (1 \leq i \leq q-1). && (30)
\end{aligned}$$

Notice that given $a_{1,k}$ $(1 \leq k \leq q-1)$, $a_{i,k}$ $(2 \leq i \leq q-1, 1 \leq k \leq q-1)$ are reshuffles of $a_{1,k}(1 \leq k \leq q-1)$ and vice versa determined by

$$a_{i,k} = a_{1,i^{-1}k \bmod q} \Longleftrightarrow a_{1,k} = a_{i,ik \bmod q} \ \ (1 \leq i, k \leq q-1). \tag{31}$$

For each $i = 1, 2, \ldots, q-1$, write $f_{j_i} := i691\sigma_5(j)\sigma_5(p-j) \bmod q$. Then $f_{j_i} = f_{p-j_i}$ $(1 \leq i \leq q-1)$ from equation (23). Define for each $i = 1, 2, \ldots, q-1$:

$$\begin{array}{lcl} S_{i,l} & := & \{k : a_{i,k} = 2l\} \ (1 \leq k \leq q, 1 \leq l \leq q_0) \\ \Longleftrightarrow & = & \{(j_{i_1}, j_{i_2}, \ldots, j_{i_l}) : \ f_{j_{i_1}} = f_{j_{i_2}} = \cdots = f_{j_{i_l}} = k\} \ (1 \leq j_{i_1} < j_{i_2} < \cdots < j_{i_l} \leq \frac{(p-1)}{2}) \\ S_{i,l} & = & \varnothing \ (1 \leq l \leq q_0) \text{ for } l > q_0. \end{array} \tag{32}$$

Since $q > p$ and since $a_{i,k}$ $(0 \leq i, k \leq q-1)$ cannot be too large even number from equations (23) and (24), a positive integer $q_0 < q-1$ exists, depending on $p$ and $q$, satisfying the last line of equation (32). It is clear from equation (32) with the aid of equation (31) that for each $l = 1, 2, \ldots, q_0$ :

$$S_{i,l} = S_{j,l} \ (1 \leq i < j \leq q-1). \tag{33}$$

For each $q \mid A(p)$ with $q > p$, we then have from equations (31) – (33) that

$$\sum_{l=0}^{q_0} \mid S_{i,l} \mid \ = q \ (1 \leq i \leq q-1). \tag{34}$$

$$\sum_{k=1}^{q-1} a_{i,k} \ = \sum_{l=1}^{q_0} 2l \mid S_{i,l} \mid = p-1 \ (1 \leq i \leq q-1). \tag{35}$$

Equation (35) reads when $q \mid A(p)$ with $q < p$ that:

$$\sum_{k=0}^{q-1} a_{i,k} = p-1 \ (1 \leq i \leq q-1). \tag{36}$$

Equations (25) – (31) readily follow from equations (23) and (24). Equation (29) is a restatement of the remainder theorem in view of equations (23), (32) and (35). Equations (34), (35) and (36) follow from the pigeonhole principle. Since we exclusively use $S_{1,l}$ $(1 \leq l \leq q_0)$ in what follows, we show the following inequality:

$$| S_{1,l-1} | > \ l \ | S_{1,l} | \ \ (2 \le l \le q_0). \tag{37}$$

To prove inequality (37) we use the second line of equation (32) for the definition of $S_{1,l}$. Consider the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by

$$\begin{aligned} \beta(j_1, j_2, \ldots, j_l) := \ & ((\beta_1(j_1), \beta_2(j_1), \ldots, \beta_{l-1}(j_1)), (\beta_1(j_2), \beta_2(j_2), \ldots, \beta_{l-1}(j_2)), \ldots, \\ & (\beta_1(j_l), \beta_2(j_l), \ldots, \beta_{l-1}(j_l))) \end{aligned} \tag{38}$$

such that for each $i = 1, 2, \ldots, l$:

$$\begin{aligned} \beta_1(j_i) \ & := \min_{j_{i_k}} \{ | \ j_i - j_{i_k} \ |: a_{1,j_{i_k}} = 2(l-1) \\ f_{\beta_1(j_i)} \ & = f_{\beta_{2(j_i)}} = \cdots = f_{\beta_{l-1}(j_i)}. \end{aligned} \tag{39}$$

In the second line of equation (39), $\beta_k(j_i)$ $(2 \le k \le l-1, 1 \le i \le l)$ are uniquely determined once $\beta_1(j_i)$ $(1 \le i \le l)$ is determined by the first line of equation (39). Observe that $(\beta_1(j_i), \beta_2(j_i), \ldots, \beta_{l-1}(j_i)) \ \in S_{1,l-1}$ $(1 \le i \le l)$ are distinct from equations (32) and (39). To show that the map $\beta$ given by equation (38) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$, write

$$\begin{aligned} \beta_1(j'_1) \ & := \min_{j_{1_k}} \{ | \ j_1 - j_{1_k} \ |: a_{1,j_{1_k}} = 2(l-1), \ \beta_1(j_1) \ne \beta_1(j'_1) \\ f_{\beta_1(j'_1)} \ & = f_{\beta_2(j'_1)} = \cdots = f_{\beta_{l-1}(j'_1)}. \end{aligned} \tag{40}$$

Observe that $(\beta_1(j'_1), \beta_2(j'_1), \ldots, \beta_{l-1}(j'_1)) \ \in S_{1,l-1}$ and distinct from $(\beta_1(j_i), \beta_2(j_i), \ldots, \beta_{l-1}(j_i))$ $(1 \le i \le l)$ from equations (32), (39) and (40). Equations (39) and (40) imply that the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by equation (38) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$ in a fashion of 1 to $l$. This establishes inequality (37). See Table 1 for examples of primes $p$ with $q \mid A(p), \ q > p$, satisfying inequality (37), where $q_0 \le 4$. Lehmer's conjecture therefore is equivalent via equation (29) for $i = 1$ to:

$$\sum_{k=0}^{q-1} k a_{1,k} \not\equiv 0 \mod q. \tag{41}$$

Since both $A(p)$ and $B(p)$ are even and divisible by 691, we have $(A(p), B(p)) \geq 1382$. Suppose $q$ divides both $A(p)$ and $B(p)$. Then by equation (29), we have:

$$\sum_{k=0}^{q-1} ka_{i,k} \equiv 0 \mod q \ (0 \leq i \leq q-1). \tag{42}$$

Clearly equation (42) is equivalent by equation (30) to:

$$\sum_{k=0}^{q-1} ka_{1,k} \equiv 0 \mod q. \tag{43}$$

Since $\sum_{k=0}^{q-1} ka_{0,k} = 0 \equiv 0 \bmod q$ by equation (26), it follows that $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1} = \{0\}$, the trivial additive group modulo $q$. Conversely, equation (42) or (43) implies both $q \mid A(p)$ and $q \mid B(p)$ by equation (29). On the other hand, since nonzero $a_{i,k}$ $(0 \leq i \leq q-1)$ is even and $\geq 2$ from equation (25), with the aid of the unique factorization theorem, equation (42) or (43) is equivalent to:

$$\min_{1 \leq i < j \leq q-1} \left(\sum_{k=0}^{q-1} ka_{i,k}, \sum_{k=0}^{q-1} ka_{j,k}\right) = 2q. \tag{44}$$

Consequently equation (42), (43) or (44) completely characterizes common odd prime factors of both $A(p)$ and $B(p)$. We thus have:

**Lemma 3** *The following conditions are equivalent:*

*(i) $q$ divides both $A(p)$ and $B(p)$.*

*(ii) $\sum_{k=0}^{q-1} ka_{i,k} \equiv 0 \mod q \ (0 \leq i \leq q-1)$.*

*(iii)* $\sum_{k=0}^{q-1} ka_{1,k} \equiv 0 \mod q$.

*(iv)* $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1} = \{0\}$*, the trivial additive group modulo* $q$*.*

*(v)* $\min\limits_{1\le i<j\le q-1}(\sum_{k=0}^{q-1} ka_{i,k},\ \sum_{k=0}^{q-1} ka_{j,k}) = 2q$.

**Lemma 4** *(Main Lemma) Let* $p$ *satisfy equation* (10) *or* (11) *and let* $q \mid A(p)$ *with* $q > p$*. Then* $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ *forms an additive group of order* $q$ *modulo* $q$*.*

**Proof.** Let $a_{i,k}$ $(0 \le i, k \le q-1)$ be defined by equation (23). We have for each $i = 1, 2, \ldots, q-1$ :

$$\begin{array}{llll} & \sum_{k=0}^{q-1} ka_{i,k} & + \quad \sum_{k=0}^{q-1} ka_{q-i,k} & \\ = & \sum_{k=1}^{q-1} ka_{i,k} & + \quad \sum_{k=1}^{q-1} ka_{i,q-k} & \text{by } (28) \\ = & \sum_{k=1}^{q-1} ka_{i,k} & + \quad \sum_{k=1}^{q-1}(q-k)a_{i,k} & \\ = & q\sum_{k=1}^{q-1} a_{i,k} & & \\ = & q\sum_{l=1}^{q_0} 2l \mid S_{1,l} \mid & \text{by } (35) & \\ = & q(p-1) & \text{by } (35) & \end{array} \tag{45}$$

Notice that equation (45) holds regardless of $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ being trivial or not. We claim that $\{\sum_{k=0}^{q-1} ka_{i,k}\}_{i=0}^{q-1}$ are all distinct. To show the claim observe that $\{S_{1,l}\}_{l=0}^{q_0}$ are disjoint from equation (32). Since $a_{i,k} = a_{1,i^{-1}k \bmod q}$ from equation (31), we have for each $l = 1, 2, \ldots, q_0$ :

$$\begin{array}{lcl} \sum\limits_{k\in S_{1,l}} ka_{i,k} & = & \sum\limits_{k\in S_{1,l}} ka_{1,i^{-1}k \bmod q} \quad = \\ \sum\limits_{k\in S_{1,l}} ik \pmod q\ a_{1,k} & = & 2l\sum\limits_{k\in S_{1,l}} ik \pmod q. \end{array} \tag{46}$$

It is evident for each $1 \le i \ne j \le q-1$ and each $l$ $(1 \le l \le q_0)$ that:

$$\sum_{k\in S_{1,l}} ik \pmod q \ne \sum_{k\in S_{1,l}} jk \pmod q. \tag{47}$$

For each $1 \leq i \neq j \leq q-1$, conjunction of equations (46) and (47) leads us to

$$\begin{array}{ll} & \sum_{k=0}^{q-1} k a_{i,k} \\ = & \sum_{l=1}^{q_0} 2l \sum\limits_{k \in S_{1,l}} ik \pmod{q} \ \text{ by (46)} \\ \neq & \sum_{l=1}^{q_0} 2l \sum\limits_{k \ in S_{1,l}} jk \pmod{q} \ \text{ by (37) \& (47)} \\ = & \sum_{k=0}^{q-1} k a_{j,k} \ \text{ by (46).} \end{array} \tag{48}$$

Equation (48) establishes the claim. Since $\sum_{k=0}^{q-1} k a_{1,k} \bmod q$ is a generator for the additive group $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1}$ from equation (30) if it is nontrivial, it suffices therefore to show that

$$\sum_{k=0}^{q-1} k a_{1,k} \not\equiv 0 \bmod q. \tag{49}$$

Write

$$C_i := \sum_{k=0}^{q-1} k a_{i,k} \ (1 \leq i \leq q-1). \tag{50}$$

Notice that $\{C_i\}_{i=1}^{q-1}$ are distinct from equation (48). Rename $C_i$ $(1 \leq i \leq q-1)$ again as $C_i$ $(1 \leq i \leq q-1)$ in ascending order as follows:

$$C_1 < C_2 < \cdots < C_{q-1}. \tag{51}$$

We claim that there is at least one pair $\{C_j, C_{j+1}\}$ $(1 \leq j \leq q-2)$ from equation (51) such that

$$C_{j+1} - C_j < q - 1 \text{ for some } j \ (1 \leq j \leq q-2). \tag{52}$$

Assume equation (52) is false. We then have:

$$\begin{array}{ll} & C_{q-1} \\ := & \max\limits_{1 \leq i \leq q-1} \sum_{k=1}^{q-1} k a_{i,k} \ \text{ by (51)} \\ := & \sum_{k=1}^{q-1} k a_{i_0,k} \ \text{ for some } \ i_0 \ (1 \leq i_0 \leq q-1) \\ = & C_1 + \sum_{k=1}^{q-2} (C_{k+1} - C_k) \\ \geq & C_1 + \sum_{k=1}^{q-2} (q-1) \ \text{ by assumption} \\ = & C_1 + (q-2)(q-1) \\ > & (q-2)(q-1). \end{array} \tag{53}$$

On the other hand, we estimate $C_{q-1}$ from equations (23) and (46). Since each nonzero $a_{i_0,k}$ $(0 \le i_0 \le q-1)$ is even $\ge 2$ from equation (25), there are at most $(p-1)/2$ -numbers of nonzero $a_{i_0,k} \ge 2$ $(0 \le k \le q-1)$. Notice that each nonzero $a_{i_0,k}$ is a small even number due to equations (32) and (35) with $2 \le a_{i_0,k} \le 2q_0$ $(0 \le k \le q-1)$. It follows that there are at least $(q-(p-1)/2)$ -numbers of $a_{i_0,k} = 0$ $(0 \le k \le q-1)$. We then have:

$$\begin{array}{ll} & C_{q-1} \\ = & \sum_{k=0}^{q-1} k a_{i_0,k} \\ = & \sum_{k=0}^{q-1} i_0 k \pmod{q}\ a_{1,k} \quad \text{by (31)} \\ = & \sum_{l=1}^{q_0} 2l(\sum\limits_{k \in S_{1,l}} i_0 k \pmod{q}) \text{ by (46)} \\ < & (\sum_{l=1}^{q_0} 2l \mid S_{1,l} \mid)(q-1) \\ = & (p-1)(q-1) \quad \text{by (35)} \\ < & (q-2)(q-1). \end{array} \tag{54}$$

In the last line of inequality (54), we use the assumption $p+1 < q$ and hence $p-1 < q-2$. The last line of inequality (54) contradicts inequality (53). This establishes inequality (52). For $j$ chosen from inequality (52), since each nonzero $a_{i,k} \ge 2$ $(1 \le i \le q-1,\ 0 \le k \le q-1)$, we then have:

$$2 \le (C_j,\ C_{j+1}) = (C_j,\ C_{j+1} - C_j) < q-1. \tag{55}$$

Equation (55) implies $C_j := \sum_{k=0}^{q-1} k a_{u,k}$ and $C_{j+1} := \sum_{k=0}^{q-1} k a_{v,k}$ for some $u, v$ $(1 \le u,\ v \le q-1)$, have no common factor $q$, which leads to $q \nmid \sum_{k=0}^{q-1} k a_{1,k}$ in view of equation (30), thereby proving equation (49). Consequently, each $\sum_{k=0}^{q-1} k a_{i,k}$ $(1 \le i \le q-1)$ has no factor $q$ from equations (30) and (49). We thus have:

$$\sum_{k=0}^{q-1} k a_{i,k} \not\equiv 0 \bmod q,\ 1 \le i \le q-1. \tag{56}$$

Equation (56) is equivalent that the map:

$$\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1} \longmapsto \mathbb{Z}/q\mathbb{Z}$$

is an isomorphism. Furthermore equations (45) and (56) reveal the structure of the additive group $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ which is nontrivial, namely

$$\sum_{k=0}^{q-1} ka_{i,k} + \sum_{k=0}^{q-1} ka_{q-i,k} \equiv\ 0 \bmod q,\ 1 \leq i \leq q-1. \tag{57}$$

Equations (56) and (57) show $\sum_{k=0}^{q-1} ka_{i,k}$ mod $q$ and $\sum_{k=0}^{q-1} ka_{q-i,k}$ mod $q$ are additive inverse to each other modulo $q$ for each $i = 1, 2, \ldots q-1$. Clearly $\sum_{k=0}^{q-1} ka_{0,k} = 0 \equiv 0 \pmod q$ is the additive identity modulo $q$ from equation (26). This completes the proof of Lemma 4.

Since $p \nmid A(p)$ from equation (12), conjunction of Lemma 3 and Lemma 4 leads us to:

**Corollary 5** *Let* $691 \mid A(p)$*. An odd prime* $q$ *divides both* $A(p)$ *and* $B(p)$ *only if* $q < p$*.*

¿From Lemma 4, we have in particular for $i = 1$ :

$$B(p) = 691 \sum_{j=1}^{p-1} \sigma_5(j)\sigma_5(p-j) \equiv \sum_{k=0}^{q-1} ka_{1,k}\ \not\equiv 0 \bmod q\ \text{ by (29) \& (56).} \tag{58}$$

Equation (58) implies $q \nmid B(p)$ and hence $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem if $691 \mid A(p)$. If $691 \nmid A(p)$, then since $691 \mid B(p)$ from equation (8), we trivially have $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem in this case too. We thus have:

**Theorem 6** $\tau(p) \neq 0$ *for each prime* $p$*.*

For $691 \mid A(p)$ and $q \mid A(p)$ with $q > p$, since $\{\sum_{k=0}^{q-1} ka_{i,k}\}_{i=0}^{q-1}$ are distinct from equation (48) and since $q \nmid \sum_{k=0}^{q-1} ka_{i,k}$ $(1 \leq i \leq q-1)$ from Lemma 4, we have $\sum_{k=0}^{q-1} ka_{i,k} = 2^s t$ $(s \geq 1,\ t = \text{odd},\ 1 \leq i \leq q-1)$, where $q \nmid t$ from Lemma 4. Since $q > p$, and since each nonzero $a_{i,k} \geq 2$ from equation (25), there is at least one $i$ $(1 \leq i \leq q-1)$ such that $\sum_{k=0}^{q-1} ka_{i,k} = 2t,\ q \nmid t$. We thus have from Lemma 1 (statement $(v)$)with the aid of unique factorization theorem:

**Corollary 7** *Suppose $p$ satisfies equation (10) or (11). Let $q \mid A(p)$ with $q > p$. Then*

$$\min_{1 \le i < j \le q-1} \left(\sum_{k=0}^{q-1} ka_{i,k},\ \sum_{k=0}^{q-1} ka_{j,k}\right) = 2.$$

Now let $\alpha \ge 2$. Then equations (10) and (11) are no longer equivalent. As in the case of $\alpha = 1$, since $A(p^\alpha) \equiv 3 \bmod p^5$ and $p^{11\alpha-1} < A(p^\alpha) < p^{11\alpha}$ from equation (8), an almost identical proof of Lemma 2 works for $\alpha \ge 2$, where in equation (14), the upper limit for the sum is replaced by $11\alpha - 1$. We thus have:

**Lemma 8** *Let $691 \mid A(p^\alpha)$ for $\alpha \ge 2$. There is at least one prime $q \mid A(p^\alpha)$ with $q > p^\alpha$.*

For $q \mid A(p^\alpha)$, construct matrix $[a_{i,k}]_{0 \le i,\, k \le q-1}$ exactly the same way as in equation (23). Then properties (25) – (31), (33) – (37) hold with $p$ replaced by $p^\alpha$. Likewise almost identical proof of Lemma 4 works for $\alpha \ge 2$. We thus have:

**Lemma 9** *Let $691 \mid A(p^\alpha)$ for $\alpha \ge 2$. Let $q \mid A(p^\alpha)$ with $q > p^\alpha$. Then $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ forms an additive group of order $q$ modulo $q$.*

In particular for $i = 1$ from Lemma 9 and equation (29), we have for $\alpha \ge 2$

$$B(p^\alpha) = 691 \sum_{j=1}^{p^\alpha - 1} \sigma_5(j)\sigma_5(p^\alpha - j) \equiv \sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \mod q. \tag{59}$$

Equation (59) implies $q \nmid B(p^\alpha)$ and hence $A(p^\alpha) \ne B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \ne 0$ by the unique factorization theorem. If $691 \nmid A(p^\alpha)$, since $691 \mid B(p^\alpha)$ from equation (8), we then trivially have $A(p^\alpha) \ne B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \ne 0$

via the unique factorization theorem in this case too. We thus have:

**Theorem 10** *$\tau(p^\alpha) \neq 0$ for each $\alpha \geq 2$.*

Finally we show that $\tau(n) \neq 0$ for any positive integer $n$.

**Theorem 11** *(Lehmer's Conjecture) $\tau(n) \neq 0$ for each $n \geq 1$.*

**Proof.** Since $\tau(1) = 1$, it suffices to prove the theorem when $n$ is composite from Theorem 6 and Theorem 10. Write

$$n = p_0^{s_0} p_1^{s_1} \dots p_u^{s_u},\ p_0 := 2,\ s_0 \geq 0,\ s_j \geq 1,\ 1 \leq j \leq u.$$

Since $\tau(n)$ is multiplicative ([ 1 : 92 – 93 ], [ 2 : 52 – 53 ], [ 4 : 122 ], [ 5 ], [ 6 ]), Theorem 11 readily follows from Theorem 6 or Theorem 10 , namely

$$\begin{array}{rcl} \tau(n) & = & \prod_{j=0}^{u} \tau(p_j^{s_j}) \\ & \neq & 0. \end{array} \tag{60}$$

This completes the proof.

Suppose for each $\alpha \geq 1$,

$$A(p^\alpha) \equiv 0 \mod 691. \tag{61}$$

Equation (61) is equivalent to:

$$p^{(\alpha+1)} \equiv 1 \mod 691 \text{ and } (p-1,\ 691) = 1. \tag{62}$$

Equation (62) implies the following periodicity theorem modulo 691:

**Theorem 12** *(periodicity modulo* 691*) Suppose* $691 \mid A(p^\alpha)$ *for* $\alpha \geq 1$*. Then we have:*

$$A(p^{\alpha+k(\alpha+1)}) \equiv 0 \mod 691, \; k = 0, 1, 2, \ldots$$

The values of $\alpha$ satisfying the periodicity of $A(p^\alpha) \equiv 0 \bmod 691$ for each $\alpha \geq 1$ have gaps in view of equation (62) and Fermat's little theorem, namely $A(p^\alpha) \not\equiv 0 \bmod 691$ if and only if the factors of $\alpha + 1$ do not divide $690 = 2.3.5.23$. Thus $A(p^\alpha) \not\equiv 0 \bmod 691$ for $\alpha$ in the following set $S$ of numbers:

$$S := \{6, 10, 12, 16, 18, 28, 30, 36, 40, 42, 46, 48, 52, 58, \ldots\}$$

Needless to say $A(p^\alpha) \neq B(p^\alpha)$ and hence $\tau(p^\alpha) \neq 0$ for each $\alpha \in S$ by equation (8) with the aid of the unique factorization theorem.

**Remark 13** *If an odd prime* $q \mid A(p^\alpha), \; \alpha \geq 1$ *with* $q < p^\alpha$*, as long as* $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1}$ *forms an additive group of order* $q$ *modulo* $q$*, then* $q \nmid B(p^\alpha)$ *by Lemma 4 or Lemma 9. It follows that* $A(p^\alpha) \neq B(p^\alpha)$ *and hence* $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ *in this case too. For* $691 \mid A(p)$*, computer simulation reveals* $A(p)$ *has at least one odd prime factor* $q \neq 691, \; q \mid A(p)$ *with* $q < p$ *for which* $q \nmid B(p)$ *for each prime* $p \leq 1100000$ *except* $p = 186569, 290219, 464351, 671651$*. Let* $691 \mid A(p)$ *and let* $A_1(p)$ *be the product of prime divisors* $q \mid A(p)$ *for which* $q < p$ *with their respective powers and* $A_2(p)$ *the product of prime divisors* $q \mid A(p)$ *for which* $q > p$ *with their respective powers. Computer simulation shows* $C_1 p^2 < A_1(p) < C_2 p^5$ *and* $C_3 p^6 < A_2(p) < C_4 p^{10}$ *with absolute constants* $C_1, C_2, C_3, C_4 < 1$ *for primes* $p \leq 1100000$*.*

In Table 1, we list primes $p$ such that both 691 and $q$ divide $A(p)$ with $q > p$ and the cardinality $\mid S_{1,l} \mid$ $(1 \leq l \leq 5)$, thereby confirming inequality (37) with $q_0 \leq 4$. Notice that in Table 1, each prime $p$ with the associated prime $q \mid A(p)$ with $q > p$, satisfies equations (34) and (35). Computer simulation reveals that the majority of respective relatively large odd prime factors less than $p$ of both $A(p)$ and $B(p)$ are distinct. Likewise an overwhelming majority of common odd prime factors of both $A(p)$ and $B(p)$ for which $691 \mid A(p)$ are relatively small apart from 691, thereby confirming Corollary 5. In Table 2, we list primes $p \leq 3000000$ such that $691 \mid A(p)$ and the odd prime factors of $(A(p),\ B(p))$ are $\geq 11$.

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**Table** 1

| p | q | $\mid S_{1,0} \mid$ | $\mid S_{1,1} \mid$ | $\mid S_{1,2} \mid$ | $\mid S_{1,3} \mid$ | $\mid S_{1,4} \mid$ | $\mid S_{1,5} \mid$ |
|---|---|---|---|---|---|---|---|
| 8291 | 216113 | 212008 | 4065 | 40 | 0 | 0 | 0 |
| 29021 | 1357091 | 1342657 | 14358 | 76 | 0 | 0 | 0 |
| 30403 | 1283839 | 1268731 | 15015 | 93 | 0 | 0 | 0 |
| 34549 | 789673 | 772578 | 16918 | 175 | 2 | 0 | 0 |
| 51133 | 112919 | 89995 | 20474 | 2267 | 174 | 9 | 0 |
| 53897 | 371549 | 345582 | 25014 | 925 | 28 | 0 | 0 |
| 96739 | 392957 | 347376 | 42917 | 2543 | 118 | 3 | 0 |

**Table** 2

| p | $(A(p),\ B(p))$ |
|---|---|
| 547271 | 2.3.11.691 |
| 610843 | 2.3.17.691 |
| 988129 | 2.3.5.13.691 |
| 1112509 | 2.3.5.23.691 |
| 1336393 | 2.3.101.691 |
| 1405493 | 2.3.113.691 |
| 1716463 | $2.3^2.23.691$ |
| 1875373 | 2.23.691 |
| 1940327 | $2^2.3^2.13.691$ |
| 2126897 | $2.3^3.19.691$ |
| 2128279 | $2^2.5.11.691$ |

| p | $(A(p),\ B(p))$ |
|---|---|
| 2161447 | $2^2.23.691$ |
| 2198761 | $2.43.691$ |
| 2447521 | $2.23.691$ |
| 2479307 | $2.23.691$ |
| 2538733 | $2.11.691$ |
| 2542879 | $2^4.3.5.23.691$ |
| 2956097 | $2.23.691$ |

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