Lagrange Inversion Without Tears (Analysis) (based on Henrici)

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Def 1: A formal Laurent series (f.L.s) is

$$\sum_{N}^{\infty} a_n x^n \quad ,$$

where a_n and x are symbols and N is an integer.

Def 2: $[x^n]f(x)$ is the coeff. of x^n in $f(x)$.

Def 3: $Res f(x)$ is the coeff. of x^{-1} in $f(x)$.

Def 4: Given a sequence f_i of f.L.s. starting at N_i, their sum $\sum f_i$ *makes sense* if the N_i are bounded below and for every n, the set $\{[x^n]f_i(x)\}$ has only finitely many non-zero terms, and then the coeff. of x^n in sum $\sum f_i$ is by definition, that sum.

Def. 5

$$cx^N \sum_{M}^{\infty} a_n x^n := \sum_{M}^{\infty} c a_n x^{n+N} :$$

(convince yourself that the rhs makes sense)

Def. 6 If $f(x) = \sum_N^{\infty} a_n x^n$ is a f.L.s., and so is $g(x)$ then

$$f(x)g(x) := \sum_{N}^{\infty} a_n x^n g(x)$$

(convince yourself that the rhs makes sense).

Def. 7

$$(\sum_{N}^{\infty} a_n x^n)' := \sum_{N}^{\infty} n a_n x^{n-1}$$

Prop. 1 (Product Rule): $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

Proof: True for $f(x) = x^m, g(x) = x^n$, and hence in general, since both sides are linear in $f(x)$, and in $g(x)$.

Prop. 2: $(f(x)^k)' = kf(x)^{k-1}f'(x)$

Proof: True for $k = 1$, and then by induction on k, for positive k and for negative k by using the product rule applied to $1 = f(x)^k f(x)^{-k}$.

Prop. 3: (Chain Rule): If Φ and $f(x)$ are f.L.s. then $(\Phi(f(x)))' = \Phi'(f(x))f'(x)$.

Proof: True for $\Phi = x^k$ thanks to Prop.2, now extend by linearity.

Prop. 4: If $f(x)$ is a f.L.s. then $Res(f'(x)) = 0$.

Proof: $(x^n)' = nx^{n-1}$ can never be a multiple of x^{-1}, hence true for monomials, and by linearity for all $f(x)$.

Prop. 5 (Integration by parts) If $f(x)$ and $g(x)$ are f.L.s. then $Res(f'(x)g(x)) = -Res(f(x)g'(x))$.

Proof: By Prop. 1 and 4.

Prop. 6 (change of variables): Let $u(t)$ be a $f.L.s$ starting at t (i.e. $N = 1$) and $\Psi(z)$ be any f.L.s. then $Res_t(u'(t)\Psi(u(t))) = Res_z\Psi(z)$

Proof: By linearity enough to prove it for monomials $\Psi(z) = z^k$. Both sides are 0 if $k \neq -1$, the right by definition 3, the left by Prop. 4. When $k = -1$ the right is 1, by definition and the left is $Res(u'(t)/u(t)) = (u_1 + 2u_2t + \ldots)/(u_1t + u_2t^2 + ...) = 1/t + O(1)$.

Theorem (Lagrange Inversion Theorem): If $u(t)$ and $\Phi(t)$ are f.L.s. starting at t and t^0 respectively, then $u(t) = t\Phi(u(t))$ implies $[t^n]u(t) = (1/n)[z^{n-1}]\Phi(z)^n$.

Proof:

$$[t^n]u(t) = Res_t(u(t)t^{-n-1}) = Res_t(u(t)(t^{-n}/(-n))') \overset{(5)}{=} (1/n)Res_t(u(t)'t^{-n}) \overset{given}{=}$$

$$(1/n)Res_t(u'(t)(\Phi(u(t))/u(t))^n) \overset{(6)}{=} (1/n)Res_z(\Phi(z)^n/z^n) = (1/n)[z^{n-1}]\Phi(z)^n \quad \square.$$

Added March 2, 2015: Generalizing, we have

Theorem (Generalized Lagrange Inversion Theorem): If $u(t)$ and $\Phi(t)$ are f.L.s. starting at t and t^0 respectively, and $G(t)$ is yet another formal power series, then $u(t) = t\Phi(u(t))$ implies $[t^n]G(u(t)) = (1/n)[z^{n-1}]\,G'(z)\Phi(z)^n$.

Proof:

$$[t^n]G(u(t)) = Res_t(G(u(t))t^{-n-1}) = Res_t(G(u(t))(t^{-n}/(-n))') \overset{(5)}{=} (1/n)Res_t(G'(u(t))u(t)'t^{-n}) \overset{given}{=}$$

$$(1/n)Res_t(u'(t)G'(u(t))(\Phi(u(t))/u(t))^n) \overset{(6)}{=} (1/n)Res_zG'(z)(\Phi(z)^n/z^n) = (1/n)[z^{n-1}]G'(z)\Phi(z)^n \quad \square.$$