The $n^{n-2}{}^{th}$ Proof Of The Formula For The Number Of Labelled Trees

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There are probably already more than 4^2 different proofs of the Cayley-Borchardt formula, n^{n-2}, for the number of labelled trees on n vertices([K],[M].) The one I present here is not the prettiest,this honor goes to Joyal's[J] (see also [L]), and not the ugliest, but it is *mine* (although it has some similarities with Clarke's proof[C].)

Let $R(e,b)$ be the number of ways of organizing e employees and b bosses, such that every employee has exactly one immediate supervisor (who may be another employee or a boss), in such a way that no employee is her own (immediate or non-immediate) superior. The ways of deciding the organization can be split into four independent decisions:

(i) How many employees, s, $1 \le s \le e$, will report directly to bosses? Let's call them *little bosses*.

(ii) Which s of the e employees will be named *little bosses*: $\binom{e}{s}$ ways.

(iii)How to organize the s little bosses and remaining $e-s$ employees amongst themselves: $R(e-s,s)$ ways.

(iv) For each of the s little bosses, which big boss should she report to?: b^s ways.

Hence:

$$R(e,b) = \sum_{s=1}^{e} \binom{e}{s} b^s R(e-s,s) \quad , \qquad (*)$$

which together with the trivial initial condition $R(0,b) = 1$, uniquely determines R. Since the recurrence (*), and the initial condition, are also satisfied by $b(b+e)^{e-1}$, by the binomial theorem, it follows that $R(e,b) = b(b+e)^{e-1}$. In particular, $R(n-1,1)$, the number of labelled trees on n vertices, equals n^{n-2}.

References

[C] L.E. Clarke, *On Cayley's formula for counting trees*, J. London Math. Soc. **33**(1958), 471-475.
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[L] G. Labelle, *Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange*, Adv. in Math. **42**(1981), 217-247.
[M] J.W. Moon, *"Counting Labelled Trees"*, Canad. Math. Congress, Montreal, 1970.

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