“Book Review of “Kepler’s Conjecture“ by George G. Szpiro“

Doron ZEILBERGER [1]

The most difficult open problems in mathematics are often very easy to state. One famous example is the Four Color Conjecture, proved in 1976 by Kenneth Appel and Wolfgang Haken (and described beautifully in Robin Wilson’s wonderful book: “Four Colors Suffice“). Another, equally famous, and even older, open problem was the Kepler conjecture, posed in 1611 by Johannes Kepler, in his correspondence with Thomas Hariot, who was wondering about the best way to stack cannon balls. Kepler conjectured that the density of a packing of congruent spheres in three dimensions is never greater than $\pi/\sqrt(18)$. It is achieved with the so-called *face-centered cubic* packing, which is just the *grocer’s method* of arranging oranges on a fruit-stand.

Unlike the Four Color Problem, that for many years was mainly attempted by amateurs, the Kepler conjecture received the attention of some of the greatest mathematicians, including Newton and Gauss. It even made it to the Pantheon of open problems of 1900, Hilbert’s famous list of 23 open problems.

Like the Four Color Problem, the ultimate solution of Kepler’s conjecture, by Thomas Hales, was computer-aided, and the computational and programming effort even exceeded that of the Four Color Theorem. Indeed such is the complexity of Hales’s proof that the paper that was submitted in 1998, is still under review, and its official status is “submitted”. Currently Thomas Hales and his students are preparing a “second-generation” proof that they hope will be more streamlined and easier to check.

The substance of the statements of the Four Color Theorem and the Kepler-Hales Theorems are very different, but the *methodologies* of proof are strikingly similar. Reduce an a priori infinite problem to checking a finite, albeit huge, number of cases. This *idea* of reduction from the infinite to the finite was made by humans (H. Heesch for Four-Color and L. Fejes-Tóth for Kepler), as was the strategy of reducing it from *finite in principle* to *feasible* (Appel-Haken and Hales respectively). But the *bottom line* required very extensive computer-checking. In the case of Hales’s proof “it takes over 270 pages of mathematical text, extensive computer resources, including 3 giga-bytes of data, well over 40 thousand lines of code, about 100000 linear programs each involving perhaps 200 variables and 1000 constraints”. It is all made mathematically rigorous by using ‘interval-arithmetic’ rather than the much faster, but non-rigorous, floating-point arithmetic.

Most mathematicians dislike and mistrust computer-assisted proofs, since they seem to violate their Platonic a priorism, and ‘contaminates’ their field with the contingencies of experimental science. But the reason mathematicians were able to manage, so far, with their tiny computer between their shoulders, is that they only proved (relatively) trivial results. What makes the proofs of the Four

[1] Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. `zeilberg@math.rutgers.edu`
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Color and the Kepler Conjectures even more significant than their already immense face-values (as proofs of long-standing extremely difficult problems), is that they are *harbingers* and *iconic examples* of future mathematics, that will be all computer-assisted, and eventually computer-generated.

George Szpiro's excellent book on Kepler's Conjecture describes very vividly, and extremely lucidly, the fascinating history of this tantalizing problem, and its ultimate proof by Thomas Hales. Along the way there are exciting digressions into many other topics, including the Four Color Problem, and philosophical discussions about computer proofs. The main text is entirely non-technical, and even the appendices require very little background. Yet everything is mathematically accurate, and the author's previous life as a professional mathematician is clear throughout.

Analogously to the Kepler and Four Color Problems, the importance of Szpiro's book transcends its already very great *face value* of telling the thrilling story of a major math problem. It is a *paradigm* of mathematical popularization, hopefully to be emulated in the future, thereby making math reach wider and wider audiences.