Spelling Out Kathy's Unit 6

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An alternative way to do Unit 6 #4

We have to prove that

If x^* is a **local maximum** of $\ln f(x)$ then it is also a local maximum of $f(x)$.

Putting $g(x) = \ln f(x)$, this is **equivalent** to the statement

If x^* is a **local maximum** of $g(x)$ then it is also a local maximum of $e^{g(x)}$.

(This is true because, of course, $f(x) = e^{\ln f(x)}$.)

But you don't need calculus for that! Pre-calculus suffices. Since the **exponential** function is an **increasing** function, $e^{g(x)}$ has the same **ups and downs** as $g(x)$.

So it is clear that $g(x)$ and $e^{g(x)}$ share their maxima (and minima!).

But, if you want to use calculus, you can.

We are given that $g'(x^*) = 0$, $g''(x^*) < 0$.

Since $f(x) = e^{g(x)}$, we have, by the **chain rule**

$$f'(x) = e^{g(x)} \cdot g'(x) \tag{1}$$

In particular

$$f'(x^*) = e^{g(x^*)} \cdot g'(x^*) = 0 \quad .$$

So we know right away that x^* is a **critical point** of $f(x)$.

To see whether it is a max or min, we need to express $f''(x)$ in terms of $g(x)$ and its derivatives.

Applying the **product rule** to Eq. (1), we have

$$f''(x) \,=\, (e^{g(x)} \cdot g'(x))' \,=\, (e^{g(x)})' \cdot g'(x) \,+\, e^{g(x)} \cdot g''(x) \tag{2}$$

Using Eq. (1) again we have

$$f''(x) \,=\, e^{g(x)} \cdot g'(x) \cdot g'(x) \,+\, e^{g(x)} \cdot g''(x) = \, e^{g(x)} \cdot g'(x)^2 \,+\, e^{g(x)} \cdot g''(x) \tag{3}$$

Factoring out $e^{g(x)}$ we finally get

$$f''(x) \,=\, e^{g(x)}(g'(x)^2 \,+ g''(x)) \quad . \tag{3}$$

Plugging-in $x = x^*$ we get

$$f''(x^*) \;=\; e^{g(x^*)}(g'(x^*)^2 \;+ g''(x^*)) \quad . \tag{4}$$

But, we already know that $g'(x^*) = 0$, so

$$f''(x^*) \;=\; e^{g(x^*)}(0^2 \;+ g''(x^*)) \;=\; e^{g(x^*)}(0 \;+ g''(x^*)) \;=\; e^{g(x^*)} \cdot g''(x^*) \quad . \tag{5}$$

Since $e^{anything}$ is always **positive**, and by assumption $g''(x^*) < 0$, and since *positive times negative is negative* , we proved that $f''(x^*) < 0$. Combined with the above fact that $f'(x^*) = 0$, this proves that x^* is also a local maximum of $f(x) = e^{g(x)}$.

Comment: No offense to calculus, the above proof using **precalculus** is much better and more insightful. To formally prove that the exponential function e^x is an **increasing function** you could of course take the derivative $(e^x)' = e^x$ and argue that it is always positive, but using **high school algebra** it is obvious that

Precalculus Lemma: If $b > a$ then $e^b > e^a$.

Proof: $b - a$ is positive hence $e^{b-a} > 1$ hence $e^b/e^a > 1$ hence $e^b > e^a$